

MODULAR FIBERS AND ILLUMINATION PROBLEMS.

P. HUBERT, M. SCHMOLL AND S. TROUBETZKOY

ABSTRACT. For a Veech surface (X, ω) , we characterize $\text{Aff}^+(X, \omega)$ invariant subspaces of X^n and prove that non-arithmetic Veech surfaces have only finitely many invariant subspaces of very particular shape (in any dimension). Among other consequences we find copies of (X, ω) embedded in the moduli-space of translation surfaces. We study illumination problems in (pre-)lattice surfaces. For (X, ω) prelattice we prove the at most countableness of points non-illuminable from any $x \in X$. Applying our results on invariant subspaces we prove the finiteness of these sets when (X, ω) is Veech.

1. INTRODUCTION

Consider a room (a plane domain) with mirror walls and a point source of light which emits rays in all directions. Is the whole room illuminated? This question is known as *the illumination problem*. The problem is attributed to Ernst Strauss in the 1950's. It has appeared in various lists of unsolved problems [Kl2, KlW, CFG]. Certain published versions of the question specify that the room has polygonal boundary [Kl1, Kl2], however the earliest published version of the question and results which we found is for rooms with smooth boundary [PP].

In this article we are interested in this question when the room is a polygon. So far, except for the trivial fact that convex rooms are illuminated by any point, all the known results on the illumination problem in polygons are negative. Tokarsky has constructed polygons P and point sources $p \in P$ which do not illumine all points $q \in P$ [To].¹ There have been various generalizations of the illumination property, for example the study of illumination by search lights [CG] and the study of the finite blocking property [Mo1, Mo2, G, HS].

Consider a rational polygon P . There is a well known procedure of unfolding P to a flat surface with conic singularities. We will state our

Date: February 2, 2008.

2000 Mathematics Subject Classification. 34C35, 40E10, 11P21, 51F99.

¹A similar construction was already evident in an earlier unpublished letter of M. Boshernitzan to H. Masur [B].

results in terms of certain classes of flat surfaces with singularities. By this unfolding procedure they hold for the polygons which unfold to them. In particular we consider two classes of surfaces, Veech surfaces and more generally prelattice surfaces. Their definition will be given below. Here we just note that all regular polygons unfold to Veech surfaces.

For surfaces we say that $p \in X$ illuminates $q \in X$ if there is a geodesic connecting p to q . Our main illumination results are

Theorem 1. *Let X be a prelattice surface. Then for any points $p \in X$, the set of points $q \in X$ which are not illuminable from p is at most countable.*

To prove Theorem 1 we establish a quantitative version of Kronecker's theorem.

Theorem 2. *Let X be a Veech surface. Then for any points $p \in X$, the set of points $q \in X$ which are not illuminable from p is finite.*

Affine homeomorphisms are real linear, i.e. they preserve geodesic segments on X , hence preserve illuminable configurations. Any two points $p, q \in X$ located in a convex, open set $\mathcal{O} \subset X$ can be connected by a straight line, consequently p and q illuminate one another if they do not belong to the *exceptional set*

$$\mathcal{E} := \{(p, q) \in X \times X : (\text{Aff}^+(X) \cdot p, \text{Aff}^+(X) \cdot q) \notin \mathcal{O} \times \mathcal{O}\}, \quad (1)$$

which is closed and $\text{Aff}^+(X)$ invariant. Here $\text{Aff}^+(X)$ is the group of affine homeomorphisms of X . Theorem 2 follows by combining Theorem 1 with a description of $\text{Aff}^+(X)$ -invariant subspaces in $X^2 := X \times X$ for a Veech surface X .

Our results on invariant subspaces of X^2 are of independent interest and will cover the first part of the paper. To state the Theorems we review the basic notions of translation surfaces, details can be found in the references [MT] and [Z1].

A *translation surface* is a compact orientable surface with an atlas such that away from finitely many points called *singularities* all transition functions are translations. Each holomorphic 1-form ω on a Riemann surface X induces a translation structure on the surface by taking natural charts:

$$\int_{z_0}^z \omega \quad z_0, z \in X \setminus Z(\omega).$$

The set $Z(\omega)$ where ω *vanishes* represents the singularities or *cone points* of the translation structure. An *Abelian differential* (X, ω) is a

pair consisting of a Riemann surface X and a holomorphic 1-form ω on X . The notion of translation surface and Abelian differential are equivalent.

The group $\mathrm{SL}_2(\mathbb{R})$ acts on a translation surface (X, ω) in the following sense. Given an element $A \in \mathrm{SL}_2(\mathbb{R})$, we can postcompose the coordinate functions of the charts of the (translation) atlas of (X, ω) by A . It is easy to see that this again yields a translation surface, denoted by $A \cdot (X, \omega)$.

We say that ϕ is an *affine homeomorphism* of a translation surface X if ϕ is a (orientation preserving) homeomorphism that is a diffeomorphism on $X \setminus Z(\omega)$, whose differential is a constant element of $\mathrm{SL}_2(\mathbb{R})$ in each chart of the atlas. The orientation preserving affine homeomorphisms of a translation surface form a group $\mathrm{Aff}^+(X, \omega)$. This group acts naturally on the surface X and diagonally on X^n : for $\phi \in \mathrm{Aff}^+(X, \omega)$

$$X^n \ni (z_1, \dots, z_n) \longmapsto \phi \circ_d (z_1, \dots, z_n) := (\phi(z_1), \dots, \phi(z_n)) \in X^n.$$

In this article we will mainly study the *diagonal action* of $\mathrm{Aff}^+(X, \omega)$.

Our starting point is the so called *Ratner Theorem* of Eskin, Marklof and Morris-Witte

Theorem [EMW] *If (X, ω) is a Veech surface and $(z_1, \dots, z_n) \in X^n$, then the closures $\overline{\mathrm{SL}(X, \omega) \circ_d (z_1, \dots, z_n)} \subset X^n$ are linear, complex-algebraic spaces.*

Let $D : \mathrm{Aff}^+(X, \omega) \rightarrow \mathrm{SL}_2(\mathbb{R})$ assign to each $\phi \in \mathrm{Aff}^+(X, \omega)$ its (constant) differential. The map D is a differential homomorphism whose range $\mathrm{SL}(X, \omega)$ is a Fuchsian group called the *Veech group* [V1] of X . Given an Abelian differential (X, ω) , there is the exact sequence

$$1 \longrightarrow \mathrm{Aut}(X, \omega) \xrightarrow{i} \mathrm{Aff}^+(X, \omega) \xrightarrow{D} \mathrm{SL}(X, \omega) \longrightarrow 1. \quad (2)$$

The kernel $\mathrm{Aut}(X, \omega) = \ker(D)$ is the group of *automorphisms* of (X, ω) , i.e. the group of biholomorphic maps of X preserving ω . In particular: $\mathrm{Aut}(X, \omega)$ is finite if $g(X) > 1$, or if (X, ω) is a torus marked in (at least) one point. Another consequence of Equation (2) is that $\mathrm{SL}(X, \omega)$ is the stabilizer of the $\mathrm{SL}_2(\mathbb{R})$ orbit of (X, ω) in the moduli space of translation surfaces.

We call a translation surface (X, ω) *Veech- or lattice surface*, if $\mathrm{SL}(X, \omega)$ is a *lattice* in $\mathrm{SL}_2(\mathbb{R})$, i.e. $\mathrm{vol}(\mathrm{SL}(X, \omega) \backslash \mathrm{SL}_2(\mathbb{R})) < \infty$.

Our first subspace result sharpens the above result of Eskin, Marklof and Morris-Witte. To begin with take an Abelian differential (X, ω)

and call a subspace $S \subset X^n$ *real-linear*, if S is locally, in natural coordinates

$$(p_1, \dots, p_n) \mapsto (z_1, \dots, z_n) := \left(\int_{p_{0,i}}^{p_i} \omega, \dots, \int_{p_{0,n}}^{p_n} \omega \right) \quad (3)$$

centered at a point $p_0 = (p_{0,1}, \dots, p_{0,n}) \in S$, defined by equations of the shape

$$a_1 z_1 + \dots + a_n z_n = 0, \text{ with } (a_1, \dots, a_n) \in \mathbb{R}^n. \quad (4)$$

Theorem 3. *Let (X, ω) be a lattice surface. Then all $\text{Aff}^+(X, \omega)$ orbit closures in X^n are real-linear, complex-algebraic spaces.*

A consequence of the real linearity of invariant subspaces is:

Theorem 4. *Each connected component $S \subset X^n$ of a 2-dimensional, invariant subspace carries a natural structure as a Veech-surface (S, ω_S) with Veech group $\text{SL}(S, \omega_S)$ commensurable to the Veech group $\text{SL}(X, \omega)$.*

Furthermore

Theorem 5. *Assume (X, ω) is a Veech surface. Then we have the following dichotomy:*

- either (X, ω) is arithmetic and there are infinitely many $\text{Aff}^+(X, \omega)$ invariant subspaces of any even dimension in X^n ,
- or (X, ω) is not arithmetic and X^n contains finitely many $\text{Aff}^+(X, \omega)$ invariant subspaces. In local coordinates (like in (3)) each invariant hypersurface satisfies an equation

$$\epsilon_1 z_1 + \dots + \epsilon_n z_n = 0$$

for some $\epsilon_1, \dots, \epsilon_n = -1, 0, 1$. Lower-dimensional, invariant subspaces are intersections of higher-dimensional ones.

Horizontal and vertical subspaces. Given a Riemann surface X call a subspace $S \subset X^2$ *horizontal* (*vertical*), if for a finite $F \subset X$

$$S = \coprod_{x \in F} \{x\} \times X \subset X^2, \quad S = \coprod_{x \in F} X \times \{x\} \subset X^2 \text{ respectively.}$$

To specify we sometimes write X^2 as $X_h \times X_v$, and think of a *horizontal* and *vertical* component. Sometimes we call vertical or horizontal subspaces simply *parallel* subspaces.

Slope of an invariant subspace in X^2 . As a further consequence of Theorem 3, a linear, connected $\Gamma \subset \text{Aff}^+(X, \omega)$ invariant subspace $S \subset X^2$ has a well defined slope. To start with S is (locally) given by an equation

$$a z_h = b z_v, \text{ with } (z_h, z_v) \in X^2 \text{ and } (a, b) \in \mathbb{R}^2.$$

We define its *slope* to be

$$\frac{a}{b} \in \mathbb{R} \cup \{\infty\}.$$

We can generalize and speak of a foliation $\mathcal{F}_\alpha(X^2)$ of X^2 , locally defined by linear equations with slope $\alpha \in \mathbb{R} \cup \{\infty\}$.

Lemma 6. *The slope is well defined for 2-dimensional $\text{Aff}^+(X, \omega)$ invariant subspaces of X^2 . Moreover the diagonal action of $\text{Aff}^+(X, \omega)$ preserves any foliation $\mathcal{F}_\alpha(X^2)$.*

A 2-dimensional $\text{Aff}^+(X, \omega)$ invariant subspace of X^2 has rational slope (see Theorem 5 for the non-arithmetic case).

Most of our results are consequences of Theorem 3 and Theorem 5.

Primitive and reduced translation surfaces. Call an Abelian differential (X, ω) *reduced*, if $\text{Aff}^+(X, \omega) \cong \text{SL}(X, \omega)$. Given a Veech surface (X, ω) , there is always a reduced Veech surface $(X_{\text{red}}, \omega_{\text{red}})$ and a covering map $\pi : X \rightarrow X_{\text{red}}$ such that

$$\pi^* \omega_{\text{red}} = \omega \quad \text{and} \quad \text{SL}(X, \omega) \subset \text{SL}(X_{\text{red}}, \omega_{\text{red}}).$$

In fact X_{red}^n describes the space of ordered n -markings on X up to translation maps, for more, see the discussion after the introduction.

A *primitive* Abelian differential (X, ω) is a translation surface which does not admit a translation map to a surface of lower genus, i.e. there is no covering map $\pi : (X, \omega) \rightarrow (Y, \alpha)$, such that $\omega = \pi^* \alpha$ and $g(X) > g(Y)$.

Off-Diagonal subspaces. Given a Riemann surface S , we have already defined the *diagonal* D_+ . If $\phi : S \rightarrow S$ is an involution, we denote its graph by

$$D_\phi := \text{graph}(\phi) := \{(x, y) \in S \times S : y = \phi(x)\}. \quad (5)$$

If (X, ω) has a unique involution ϕ , in particular if (X, ω) is reduced we simply write D_- instead of D_ϕ and call it the *off-diagonal*.

The next Theorem classifies all possible invariant subspaces under certain assumptions.

Theorem 7. *Let (X, ω) be a primitive Veech surface with exactly one cone-point of prime order. Then all connected components of 2-dimensional $\text{SL}(X, \omega)$ -invariant subspaces $S \subset X_{\text{red}}^2$ admit a natural translation structure (S, ω_S) , with an affine diffeomorphism*

$$(S, \omega_S) \cong (X, \omega).$$

Furthermore if (X, ω) is not arithmetic, then the only 2-dimensional, linear subspaces in X_{red}^2 away from the parallel ones are D_+ and D_- . The last subspace exists if and only if (X, ω) admits an (affine) involution.

Since X_{red}^n parametrizes n -tuples of marked points on (X, ω) , it is a fiber

$$X_{red}^n \hookrightarrow \mathrm{SL}_2(\mathbb{R}) \circ_a X_{red}^n \xrightarrow{\pi} \mathrm{SL}_2(\mathbb{R}) \cdot (X, \omega) \cong \mathrm{SL}(X, \omega) \backslash \mathrm{SL}_2(\mathbb{R}),$$

i.e. $\pi^{-1}((X, \omega)) = X_{red}^n$, in the moduli space of n -tuple markings on deformations of (X, ω) , thus we call X_{red}^n the *modular fiber*. Our results say, the modular fiber and hence moduli space itself contains Veech surfaces (S, ω_S) with a lattice group $\mathrm{SL}(S, \omega_S) \cong \mathrm{SL}(X, \omega)$. In general Veech surfaces $(S, \omega_S) \subset X^2$ are not isomorphic to the surface (X, ω) defining the modular fiber.

More illumination. Given a covering map $\pi : X \rightarrow Y$ we define $\pi_2 := \pi \times \pi : X^2 \rightarrow Y^2$ by $\pi_2(x_1, x_2) := (\pi(x_1), \pi(x_2))$. For the following we need the coordinate projection $pr_h : X^2 \cong X_h \times X_v \rightarrow X_h$, $(p, q) \mapsto p$.

Theorem 8. *Take a Veech surface (X, ω) with canonical covering $\pi : X \rightarrow X_{red}$. Assume that X_{red}^2 contains exactly D_+ and $D_- = D_\phi$, $\phi \in \mathrm{Aff}^+(X_{red}, \omega_{red})$ the (unique) involution, as non-parallel invariant subspaces. Then the only non-illuminable configurations $(p, q) \in X^2$ on X are*

- all regular pairs in $\pi_2^{-1}(D_-)$, if and only if $pr_h(\pi_2^{-1}(D_+ \cap D_-)) \subset X$ are cone points and
- some pairs $(p, q) \in X^2$ of periodic points.

In the later one needs to check for illuminability case by case.

Corollary 9. *The previous Theorem applies to all known Veech surfaces in genus 2, 3 and 4 with precisely one cone point.*

For these examples see C.T.McMullen's papers [McM1, McM3].

Remark: One can check that there are no nonillumination configurations in the nonarithmetic genus 2 (one cone point) Veech case by looking at L-shaped representatives and using Mller's classification of periodic points.

2. BACKGROUND

For later use we recall some facts on Veech surfaces (X, ω) and the flat geometry on X induced by ω .

A *saddle connection* on (X, ω) is a geodesic segment, with respect to

the flat metric induced by the translation structure, starting and ending at zeros of ω . A direction on a translation surface is called *periodic* if the flow in this direction is periodic (except for the set of saddle connections in this direction). Thus in a periodic direction the surface decomposes into maximal cylinders bounded by saddle connections. The *modulus* of a cylinder is its width divided by its length.

An element $A \in \mathrm{SL}_2(\mathbb{R}) \setminus \{\pm 1\}$ is either parabolic, elliptic, or hyperbolic. A direction on a translation surface is called *parabolic* if there is an affine diffeomorphism that preserves the set of geodesics in this direction and whose differential is parabolic. Veech has shown that a direction is parabolic if and only if it is periodic with all moduli commensurable [V1]. In this case the action of a parabolic diffeomorphism restricted to a cylinder is a power of a Dehn twist.

Reduced surfaces. As indicated by the exact sequence (2) of groups, we cannot distinguish the marked surface (X, m, ω) from the marked surface $(X, \phi(m), \omega)$ for any $\phi \in \mathrm{Aut}(X, \omega)$, at least if we consider the *moduli space* of marked points (equivalence up to translations of (X, ω)). The moduli space of marked points is given by the quotient $X_{red} := X / \mathrm{Aut}(X, \omega)$. If $\pi : X \rightarrow X_{red}$ is the quotient map we can define the (holomorphic) one form

$$\omega_{red} := \pi_* \omega$$

on X_{red} , since all automorphisms preserve ω . By definition the Abelian differential (X_{red}, ω_{red}) has no (non-trivial) automorphisms and consequently

$$\mathrm{Aff}^+(X_{red}, \omega_{red}) \cong \mathrm{SL}(X_{red}, \omega_{red}), \quad (6)$$

as well as

$$\mathrm{SL}(X, \omega) \subset \mathrm{SL}(X_{red}, \omega_{red}) \quad (7)$$

if every affine map of (X, ω) descends to an affine map of (X_{red}, ω_{red}) . To see this take $\psi \in \mathrm{Aff}^+(X, \omega)$, then $\psi \cdot f \cdot \psi^{-1} \in \mathrm{Aut}(X, \omega)$ for all $f \in \mathrm{Aut}(X, \omega)$ and consequently

$$\pi \circ \psi \circ f(x) = \pi \circ \psi \circ f \circ \psi^{-1} \circ \psi(x) = \pi \circ \psi(x),$$

showing that ψ descends. Obviously we have

$$[\mathrm{SL}(X_{red}, \omega_{red}) : \mathrm{SL}(X, \omega)] < \infty. \quad (8)$$

A useful observation is

Proposition 10. *Given a lattice surface (X, ω) with an involution $\phi \in \mathrm{Aff}^+(X, \omega)$ and assume $\mathrm{Aff}^+(X, \omega) \cong \mathrm{SL}(X, \omega)$. Then ϕ is the only affine involution of (X, ω) .*

Proof. For any affine involution $\phi \in \text{Aff}^+(X, \omega)$ we have $D\phi = -1 \in \text{SL}(X, \omega)$. By assumption $D : \text{Aff}^+(X, \omega) \rightarrow \text{SL}(X, \omega)$ is an isomorphism. \square

Because the modular fibers X_{red}^n of a lattice surface (X, ω) are always reduced and Equation (7) holds, we can assume all lattice surfaces we are looking at are reduced.

Example: Translation surfaces obtained from regular $2n$ -gons.

Veech showed in [V1, V2], that one obtains a lattice surface (Y, τ) taking 2 copies of the regular $2n$ -gon and identify sides of the first regular $2n$ -gon with the diametrical, parallel sides of the copy. Using just one $2n$ -gon and the same diametrical gluing scheme, one obtains a translation surface (X, ω) as well. Exchanging the two $2n$ -gons tiling (Y, τ) defines a nontrivial automorphism ϕ of order 2 in $\text{Aut}(Y, \tau)$. Moreover the quotient of (Y, τ) with respect to $\text{Aut}(Y, \tau) = \langle \phi \rangle$ is (X, ω) and thus $(Y_{red}, \tau_{red}) = (X, \omega)$.

We remark that all presently known primitive Veech surfaces of genus $g \geq 2$ have one or two cone points, moreover a new result of Martin Mller [M2] shows that the number of (algebraically) primitive Veech surfaces with two cone points in fixed genus is finite.

Note, that primitive (lattice) differentials are reduced, but a reduced lattice differential is not necessary primitive. An infinite set of reduced elliptic differentials (X, ω) with $\text{SL}(X, \omega) \cong \text{SL}_2(\mathbb{Z})$ is contained in [S2, S3]. Square-tiled surfaces in $\mathcal{H}(2)$ [HL] provide examples of reduced, but not primitive arithmetic Veech surfaces too.

3. INVARIANT SUBSPACES OF X^n

The zero dimensional $\text{SL}(X, \omega)$ invariant subspaces of X^n are periodic points. Each point (Y, τ) having finite $\text{Aff}^+(X, \omega)$ orbit corresponds to (X, ω) marked at some periodic points (m_1, \dots, m_n) . For reduced (X, ω) ,

$$[\text{SL}(X, \omega) : \text{SL}(Y, \tau)] = |\text{SL}(X, \omega) \cdot [Y, \tau]|.$$

If (X, ω) is *not* an elliptic differential, i.e. not a torus cover, a result of Gutkin, Hubert and Schmidt [GHS] says there are only *finitely many* periodic orbits in (X, ω) and thus for all n , there are only finitely many periodic points in X^n .

Reality of invariant subspaces and consequences. If a point

$z = (z_h, z_v) \in X^2$ is not periodic, at least one of its components, say z_h , is not periodic and thus has closure $X = \overline{\text{SL}(X, \omega)z_h}$ by [GHS]. Assume S is a connected $\text{SL}(X, \omega)$ invariant subset of X^2 . Ratner's Theorem says that $\text{SL}(X, \omega)$ invariant subsets in X^2 are already linear, complex submanifolds. We also have

Proposition 11. *A point $p \in X^2$ is periodic with respect to the action of $\text{SL}(X, \omega)$, if and only if $p \in L \cap S$ is in the intersection of two connected, invariant subspaces $L \neq S \subset X^2$ with $\dim(L) = \dim(S) = 2$.*

Proof. If $L, S \subset X^2$ are two connected, linear, invariant subspaces of complex dimension 1, they have only finitely many intersection points or $L = S$. By invariance of L and S , $L \cap S$ is also invariant under the action of $\text{Aff}^+(X, \omega)$. If $p = (p_h, p_v) \in X \times X$ is periodic, both components p_i are periodic and we have

$$p \in \{p_h\} \times X \cap X \times \{p_v\}.$$

□

From now on we call a subspace $S \subset X^n$ *invariant*, if it is a connected component, or a union of connected components of an $\text{SL}(X, \omega)$ -invariant set. Let

$$I = I_m := \{i_1, \dots, i_m\} \subset \{1, \dots, n\} \quad (9)$$

be a tuple and

$$\text{pr}_I : X^n \rightarrow X^m = X_{i_1} \times X_{i_2} \times \dots \times X_{i_m}$$

the canonical projection. Then we have

Proposition 12. *Assume (X, ω) is a lattice surface and $S \subset X^n$ is invariant, then*

- *intersections of invariant subsets of X^n are invariant and*
- *the sets $\text{pr}_{I_m}(S)$ and $\text{pr}_{I_m}^{-1} \circ \text{pr}_{I_m}(S)$ are invariant for each index subset I_m .*

Proof. The first claim is obvious. For the second claim, we note that $\text{pr}_{I_m}(S)$ is $\text{SL}(X, \omega)$ equivariant. □

Proof of Theorem 3 for X^2 . This argument proves the reality of invariant subspaces $S \subset X^2$ of dimension 2. We can restrict ourselves to invariant subspaces S , which are $\text{Aff}^+(X, \omega)$ orbit closures, i.e. S with a finite number of connected components.

From [EMW], see Ratner's Theorem we know that any invariant subset $S \subset X^2$ which is an $\text{Aff}^+(X, \omega)$ orbit-closure of one point $z \in X$

is linear, algebraic and of even dimension. In particular S has finitely many connected components and is stabilized by a finite index subgroup $\Gamma_S \subset \text{SL}(X, \omega)$.

The statement clearly holds for horizontal and vertical subspaces S of X^2 . Thus we assume $S \subset X^2$ is a connected subspace which is Γ_S invariant for a finite index subgroup of $\text{SL}(X, \omega)$ and not horizontal or vertical. In this case the two projections

$$\pi_i : X^2 \rightarrow X_i \cong X, \quad i = h, v$$

are onto and S must be a linear *correspondence*, i.e. both maps $\pi_i : S \rightarrow X_i$ are finite to one. By Γ_S invariance and linearity we might describe S in a neighborhood of $p \in S$ as $(y, \psi(y)) \in X^2$ with an affine map $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. After application of $\phi \in \text{Aff}^+(X, \omega) \cap D^{-1} \Gamma_S$ we get by invariance

$$(\phi(y), \phi \cdot \psi(y)) = (z, \psi_\phi(z)) \in S,$$

with an affine map $\psi_\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. But $z = \phi(y)$ and thus:

$$(\phi(y), \phi \cdot \psi(y)) = (z, \psi_\phi(z)) = (\phi(y), \psi_\phi \cdot \phi(y)). \quad (10)$$

Now for all $\phi \in \text{Aff}^+(X, \omega) \cap D^{-1} \Gamma_S$ and all pairs (ψ, ψ_ϕ) we obtain by linearity and connectedness of S :

$$D\psi = D\psi_\phi =: B.$$

Thus taking the linear part in the second component of Equation (10) gives

$$A \cdot B = B \cdot A \quad \text{for all } A \in \Gamma_S.$$

Without restrictions we assume that the vertical and horizontal foliation on X is periodic. Taking the subgroup of Γ_S generated by the Dehn twists along the horizontal and vertical direction proves:

$$B = b \cdot \text{id} \in \text{GL}_2(\mathbb{R}) \quad \text{with } b \in \mathbb{R} - \{0\}.$$

That means all linear, connected Γ_S invariant $S \subset X^2$ of dimension 2 are algebraic submanifolds of X^2 defined by an linear equation with real coefficients. \square

Theorem 3 allows to identify directions on horizontal and vertical fibers in X^2 via any $\text{SL}(X, \omega)$ invariant subspace S . We can, for example, take a closed leaf $\gamma \subset (X, \omega)$ viewed as a horizontal embedding

$$\gamma \subset (X, \omega) \rightarrow X \times \{z_v\} \subset X^2.$$

Then an invariant subspace S maps the image of $\gamma \subset X \times \{z_v\}$ to a closed leaf $\gamma_S \subset \{z_h\} \times X$ in the same direction for any $z_h \in X$.

Proof of Lemma 6. Because $\text{Aff}^+(X, \omega)$ acts diagonally on X^2 and real linear on each component of X^2 , we find for any $\phi \in \text{Aff}^+(X, \omega)$

$$a\phi(z_h) - b\phi(z_v) = D\phi(az_h - bz_v) + c_\phi, \text{ with } a, b \in \mathbb{R}, c_\phi \in \mathbb{R}^2.$$

This implies the slope is well defined for 2-dimensional $\text{Aff}^+(X, \omega)$ invariant subspaces of X^2 and the diagonal action of $\text{Aff}^+(X, \omega)$ preserves any foliation $\mathcal{F}_\alpha(X^2)$. \square

Canonical holomorphic differential on invariant subspaces. A further consequence of the real-linearity of an $\text{SL}(X, \omega)$ invariant subspace S is that it admits a holomorphic 1-form ω_S , compatible with the 1-form ω on X with respect to both coordinate projections $\pi_i : S \rightarrow X_i$. In fact locally and away from its zeros ω_S might be defined by

$$dz_S := \alpha\sqrt{1+\alpha^{-2}} dz_h = \alpha^{-1}\sqrt{1+\alpha^2} dz_v \quad (11)$$

for an invariant S with slope $\alpha \neq 0, \infty$. If S is horizontal ($\alpha = 0$) or vertical ($\alpha = \infty$) we have $dz_S = dz_h, dz_S = dz_v$ respectively.

Cone points of (S, ω_S) . To characterize the cone points of (S, ω_S) and prove finiteness of invariant subspaces, we note

Proposition 13. *Let (S, ω_S) be an $\text{Aff}^+(X, \omega)$ invariant subset. Assume one of the coordinates $(z_h, z_v) \in S \subset X^2$ is periodic in X (with respect to the action of $\text{Aff}^+(X, \omega)$). Then either (S, ω_S) is vertical, horizontal, or (S, ω_S) has nontrivial slope and both coordinates, z_h and z_v are periodic.*

Proof. Assume $z_h \in X$ is periodic and z_v is not, then

$$\overline{\text{Aff}^+(X, \omega) \cdot (z_h, z_v)} = \coprod_{z \in \text{Aff}^+(X, \omega) \cdot z_h} \{z\} \times X$$

is horizontal. If z_v is periodic, but not z_h we obtain the analogous statement with a 'vertical' subspace. Since a 2-dimensional, invariant subspace S with nontrivial slope cannot contain a vertical or horizontal subspace, $(z_h, z_v) \in S$ is periodic, or both coordinates z_h and z_v are non-periodic. \square

There is not much to say about the cone points of horizontal or vertical subspaces, since these spaces are isomorphic to a finite disjoint union of copies of (X, ω) .

Proposition 14. *Assume (S, ω_S) ($S \subset X^2$) is invariant, has nontrivial slope and $p = (p_h, p_v) \in S$ is periodic then*

$$o_p(\omega_S) = \text{lcm}(o_{p_h}(\omega), o_{p_v}(\omega)) \quad (12)$$

where $o_{p_h} = \text{ord}(p_h) + 1$ (and o_{p_v} is defined similarly).

Proof. We consider a full loop $\gamma \subset S$ around $p = (p_h, p_v) \in S$. Then γ projects to two loops $\gamma_h \subset X_h$, $\gamma_v \subset X_v$ around p_h, p_v respectively. The total angle along these two loops is a multiple of $o_{p_h}(\omega)$, $o_{p_v}(\omega)$ respectively. Thus the minimal total angle needed to return to the points $l_h \in \gamma_h$ and $l_v \in \gamma_v$ on the projected loops, while at the same time returning to $(l_h, l_v) \in \gamma$, must be $\text{lcm}(o_{p_h}(\omega), o_{p_v}(\omega))$. \square

4. GEOMETRY OF INVARIANT SUBSPACES.

We give some examples of invariant subspaces in the arithmetic case.

Given an invariant subspace $S \subset X_h \times X_v$ of slope $\alpha \in \mathbb{R} - \{0\}$, we assume in the following, that all images $\text{pr}_h(\mathcal{C}_{\omega_S}) \subset X_h$ and $\text{pr}_v(\mathcal{C}_{\omega_S}) \subset X_v$ of the set of cone-points \mathcal{C}_{ω_S} of (S, ω_S) are *marked*. The leaves contained in a foliation of any subspace S induce *affine maps* of leaves

$$\begin{array}{ccccc} \mathcal{F}_\theta(X_h) & \xleftarrow{\text{pr}_{h*}} & \mathcal{F}_\theta(S) & \xrightarrow{\text{pr}_{v*}} & \mathcal{F}_\theta(X_v) \\ \mathcal{L}_h & \longmapsto & \mathcal{L} & \longmapsto & \mathcal{L}_v \end{array} \quad (13)$$

which locally is a stretch by α , because S has slope α . It turns out that the interesting sub-spaces have rational slopes.

Proposition 15. *Assume $S \in \mathcal{F}_\alpha(X^2)$, with $\alpha = p/q \in \mathbb{Q}$, then the length of two compact, corresponding leaves $\mathcal{L}_h \in \mathcal{F}_\theta(X_h)$ and $\mathcal{L}_v \in \mathcal{F}_\theta(X_v)$ are related by:*

$$p \cdot |\mathcal{L}_h| = q \cdot |\mathcal{L}_v|, \quad \text{for singular leaves (saddle connections)} \quad (14)$$

and there exist $a, b \in \mathbb{N}$

$$ap \cdot |\mathcal{L}_v| = bq \cdot |\mathcal{L}_h|, \quad (a, b) = 1 \quad \text{for compact regular leaves.} \quad (15)$$

Proof. The proof uses the length induced by the 1-forms on S and X , i.e.

$$|\mathcal{L}| = \sqrt{1 + \alpha^2} \cdot |\mathcal{L}_h| = \frac{\sqrt{1 + \alpha^2}}{|\alpha|} \cdot |\mathcal{L}_v| \quad \text{or,} \quad \frac{|\mathcal{L}_v|}{|\mathcal{L}_h|} = |\alpha|. \quad (16)$$

The condition $(a, b) = 1$ comes from the fact that one completes one loop around $\mathcal{L} \subset S$, starting let's say at $(x_h, x_v) \in \mathcal{L} \subset S$, if and only if one returns to $x_h \in \mathcal{L}_h = \text{pr}_{h*} \mathcal{L}$ and $x_v \in \mathcal{L}_v = \text{pr}_{v*} \mathcal{L}$ simultaneously while walking along \mathcal{L} . \square

Characterization of cone points in leaves of $\mathcal{F}_\alpha(\mathbf{X}^2)$. Again we assume (X, ω) is a lattice surface. To describe $\text{SL}(X, \omega)$ invariant subspaces of slope $\alpha \in S^1$ in X^2 is equivalent to finding the compact leaves of the foliation $\mathcal{F}_\alpha(X^2)$. Here we study the neighborhood of a cone point $p = (p_h, p_v) \in X^2$, and the leaves of $\mathcal{F}_\alpha(X^2)$ containing the cone point p .

Two points $p_h, p_v \in X$ define a *cone point* $p = (p_h, p_v)$ of X^2 , if at least one of the two points is a cone point of (X, ω) . To study this cone point as a cone point of leaves in $\mathcal{F}_\alpha(X^2)$ we take a small circle $\gamma_r \subset S$ of radius $r > 0$ centered at p , S a leaf in $\mathcal{F}_\alpha(X^2)$ containing p , such that $p \in S$. The projections $\gamma_{r_h, \alpha} = \pi_h(\gamma_r) \subset X_h$ and $\gamma_{r_v, \alpha} = \pi_v(\gamma_r) \subset X_v$ of γ_r are circles of radius $r_h = r/\sqrt{1 + \alpha^2}$, $r_v = r \cdot |\alpha|/\sqrt{1 + \alpha^2}$ respectively centered at p_h and p_v . The product of the loops $\gamma_{r_h, \alpha}$ and $\gamma_{r_v, \alpha}$ is a torus

$$\mathcal{T}_{r, p, \alpha} := \gamma_{r_h, \alpha} \times \gamma_{r_v, \alpha} \subset X_h \times X_v.$$

We need to see how many leaves $S \subset \mathcal{F}_\alpha(X^2)$ terminate at p and intersect the torus $\mathcal{T}_{p, r, \alpha}$. A natural parametrization of $\mathcal{T}_{p, r, \alpha}$, by means of the angle around p_h and p_v is $[0, o_{p_h}] \times [0, o_{p_v}]$. In these coordinates with $*$ = h or v

$$\mathcal{F}_\theta(X) \cap \gamma_{r*} = \{\theta + i(o_{p*}) : i = 1, \dots, o_{p*}\}.$$

Consider the subspace $\mathcal{U}_{\epsilon, p, \alpha} \subset X^2$ given by

$$\mathcal{U}_{\epsilon, p, \alpha} = \{p\} \cup \bigcup_{0 < r < \epsilon} \mathcal{T}_{p, r, \alpha}.$$

Geometrically it is a solid torus where the center loop is collapsed into one point, p . It intersects all leaves of $\mathcal{F}_\alpha(X^2)$ which contain p . We have an induced foliation

$$\mathcal{F}_\alpha(\mathcal{U}_{\epsilon, p, \alpha}) := \mathcal{U}_{\epsilon, p, \alpha} \cap \mathcal{F}_\alpha(X^2).$$

The leaves of $\mathcal{F}_\alpha(\mathcal{U}_{\epsilon, p, \alpha})$ containing p are represented by

$$S_{i_h, i_v}^\epsilon := \{(r_h \exp(2\pi i(\theta + i_h)/o_{p_h}), r_v \exp(2\pi i(\theta + i_v)/o_{p_v})) : \theta \in \mathbb{R}, 0 \leq r < \epsilon\} \subset \mathcal{F}_\alpha(\mathcal{U}_{\epsilon, p, \alpha}), \quad (17)$$

with $i_h = 1, \dots, o_{p_h}$ and $i_v = 1, \dots, o_{p_v}$.

To see how many of the $o_{p_h} \cdot o_{p_v}$ leaves S_{i_h, i_v}^ϵ are really different we derive from the representation (17) that

$$S_{i'_h, i'_v}^\epsilon = S_{i_h, i_v}^\epsilon$$

whenever there is a $\theta \in \mathbb{Z}$, such that

$$i'_h \equiv \theta + i_h \pmod{o_{p_h}} \quad \text{and} \quad i'_v \equiv \theta + i_v \pmod{o_{p_v}}.$$

Thus a necessary condition for the two local leaves to be equal is

$$i_p := i_h - i_v \equiv i'_h - i'_v \pmod{\gcd(o_{p_h}, o_{p_v})} \quad (18)$$

and given $i_p \pmod{\gcd(o_{p_h}, o_{p_v})}$, there are $\text{lcm}(o_{p_h}, o_{p_v})$ leaves S_{i_h, i_v}^ϵ such that $i_h - i_v \equiv i_p \pmod{\gcd(o_{p_h}, o_{p_v})}$.

Lemma 16. *Given two cone points $p_h, p_v \in X$. Then for every $\alpha \in \mathbb{R} - \{0\}$ the point $p = (p_h, p_v) \in X^2$ defines*

$$\frac{o_{p_h} \cdot o_{p_v}}{o_p} = \gcd(o_{p_h}, o_{p_v}) \quad (19)$$

cone points of order $o_p = \text{lcm}(o_{p_h}, o_{p_v})$ contained in leaves of $\mathcal{F}_\alpha(X^2)$. In particular for small ϵ there are $\gcd(o_{p_h}, o_{p_v})$ leaves $S_{i_h, i_v}^\epsilon \in \mathcal{F}_\alpha(\mathcal{U}_{\epsilon, p, \alpha})$.

Proof. Recall that $o_p = \text{lcm}(o_{p_h}, o_{p_v})$ and note that the different cone points are defined as cone points of the local leaves S_{i_h, i_v}^ϵ containing them. Algebraically each cone point is characterized by the image of the map

$$\begin{array}{ccccc} \mathbb{Z}/o_p\mathbb{Z} & \rightarrow & \mathbb{Z}/o_{p_h}\mathbb{Z} \oplus \mathbb{Z}/o_{p_v}\mathbb{Z} & \rightarrow & \mathbb{Z}/\gcd(o_{p_h}, o_{p_v})\mathbb{Z} \\ i & \mapsto & (i_h, i_v) := (i(o_{p_h}), i(o_{p_v})) & \mapsto & i_h - i_v \pmod{\gcd(o_{p_h}, o_{p_v})}. \end{array}$$

Since for any given $i_p \in \mathbb{Z}/\gcd(o_{p_h}, o_{p_v})\mathbb{Z}$ there is a leaf S_{i_h, i_v}^ϵ with $i_p \equiv i_h - i_v \pmod{\gcd(o_{p_h}, o_{p_v})}$ and

$$o_h \cdot o_v = \text{lcm}(o_{p_h}, o_{p_v}) \gcd(o_{p_h}, o_{p_v})$$

the statement follows. \square

Alternatively one can characterize a cone point $p = (p_h, p_v) \in S \subset X^2$ by using the $o_h \cdot o_v$ images $\mathcal{L}_{i,h} := \text{pr}_h(\mathcal{L}_i) \subset \mathcal{F}_h(X)$ and $\mathcal{L}_{i,v} := \text{pr}_v(\mathcal{L}_i) \subset \mathcal{F}_v(X)$ of the horizontal leaves $\mathcal{L}_i \subset \mathcal{F}_h(S)$ emanating from p . For example the diagonal $D_+ = \{(x, x) \in X^2 : x \in X\}$ contains the cone point $p \in Z(\omega_{D_+})$, with singular leaves $\mathcal{L}_i, i = 1, \dots, o_p$ projecting to pairs of the shape $(\mathcal{L}_{i,h}, \mathcal{L}_{i,v})$.

To give an example of an invariant surface $D_2 \subset \mathcal{F}_1(X^2)$, $D_2 \neq D_+$ containing the cone point $p = (p_h, p_v) \in X^2$ ($p_h = p_v$ as cone point of X), we need to establish a correspondence of horizontal saddle connections starting at p_h and p_v in a cyclic way. To obtain a compact surface $D_2 \subset \mathcal{F}_1(X^2)$, the easiest condition to impose on X is all horizontal (and vertical) saddle connections have the same length. One arithmetic surface with this property is L_3 , the L -shaped surface in genus 2, tiled by three unit squares.

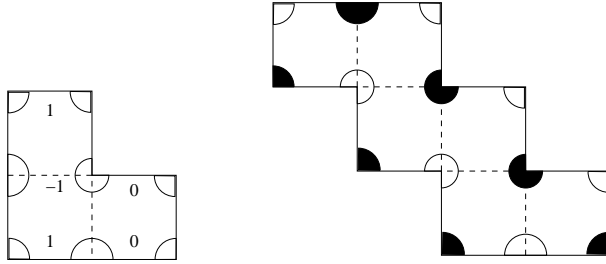


FIGURE 1. Slope 1 invariant subspaces of L_3^2 through p

Note: all horizontal and vertical relative periods of D_2 must have length 1, by the cyclic identification scheme of their projections to $L_3 = L_{3,h} = L_{3,v}$. The *staircase* shaped surface D_2 has an obvious automorphism ϕ of order 3, which is the number of *steps* in the staircase. The map ϕ might be defined by moving each step of the staircase one step up. The quotient surface $D_2 / \langle \phi \rangle$ is a 2-marked torus with Veech-group Γ_2 .

Theorem 17. *All 2-dimensional $\mathrm{SL}(L_3, \omega_{L_3})$ -invariant subspaces of $S \subset L_3 \times L_3$ containing the cone point $p = (p_h, p_v) \in L_3^2$ are Veech surfaces (S, ω_S) with*

$$\mathrm{SL}(L_3, \omega_{L_3}) \cong \Gamma_2 \subseteq \mathrm{SL}(S, \omega_S).$$

In particular $L_3 \times L_3$ contains infinitely many different arithmetic surfaces, whose Veech group contains the congruence group Γ_2 .

Proof. The relative period lattice of L_3 defines a map

$$\pi : \mathrm{SL}_2(\mathbb{Z}) \circ_d L_3^2 \longrightarrow \mathbb{T}^2 \times \mathbb{T}^2 \cong \mathbb{T}^4. \quad (20)$$

By construction π is $\mathrm{SL}_2(\mathbb{Z})$ equivariant. Since the Veech-group of L_3 is Γ_2 , $\mathrm{SL}_2(\mathbb{Z}) \circ_d L_3^2$ has 3 connected components and $L_3^2 \subset \mathrm{SL}_2(\mathbb{Z}) \circ_d L_3^2$ as well as $p = (p_h, p_v) \in L_3^2$ is stabilized by Γ_2 . By $\mathrm{SL}_2(\mathbb{Z})$ equivariance of π all 2-dimensional, $\mathrm{SL}_2(\mathbb{Z})$ -invariant subspaces in \mathbb{T}^4 have Γ_2 invariant preimages in L_3^2 . Theorem 4 says that each connected component $(S, \omega_S) \subset L_3^2$ of an invariant subspace is stabilized by a group commensurable to Γ_2 .

By the discussion before Lemma 16, there are exactly 3 local leaves containing the point $p = (p_h, p_v)$. On each of these local leaves p represents a cone point of order 3. Take the horizontal foliation on L_3 and enumerate the 3 horizontal leaves emanating from the cone point $p_h = p_v \in L_3$ by $(-1, 0, 1)$ (see enumeration in figure 2), then the three

cone points contained in the local leaves of slope 1 in L_3^2 are characterized by the identification *types* $0 \leftrightarrow 0$, $0 \leftrightarrow 1$ and $0 \leftrightarrow -1$.

Certainly the leaf containing the cone point given by $0 \leftrightarrow 0$, i.e. the trivial identification, is stable under operation of Γ_2 , because this cone point does not change type under the diagonal action of Γ_2 . The other two cone points are getting exchanged under the involution ϕ of L_3 , because ϕ changes the order of outgoing leaves, i.e. $\phi(-1, 0, 1) = (1, 0, -1)$.

Claim: the invariant subspace containing these two cone points is always connected. To see this note that any invariant subspace S has rational slope, say p/q ($p, q = 1$). Walking on a horizontal leaf emanating from $(p_h, p_v) \in L_3^2$ on S , we will hit the cone point $(p_h, p_v) \in L_3^2$ again, since S is a Veech surface. Choosing a continuation of this saddle connection gives a path γ which is a chain of saddle connections and projects to a chain of q (horizontal) saddle connections of length 1 on $L_3 \cong L_3 \times \{p_v\}$, respectively a chain of p saddle connections of the same length on $L_3 \cong \{p_h\} \times L_3$. This implies the middle point of $\gamma \subset S$ is either of the shape (p_h, w) , (w, p_v) , or of the shape (w, w) , w a (regular) Weierstrass point. But any point of the above shape is fixed under the involution ϕ . This proves connectedness of S . As a consequence we have for the Veech-group of (S, ω_S) :

$$\Gamma_2 \subseteq \mathrm{SL}(S, \omega_S).$$

□

Remark. The leaf $\mathcal{L}_2 \subset \mathcal{F}_2(L_3^2)$ containing the standard neighborhood of the cone point $p \in L_3^2$ is tiled by 36 squares, hence contains 12 copies of L . It is easy to see that \mathcal{L}_2 contains all 3 cone points defined by $p = (p_h, p_v) \in X^2$. Also note, that two cone points are exchanged by the (only) involution $\phi \in \mathrm{Aff}^+(L_3, \omega_{L_3})$.

Questions. Describe the possible subspaces of X^2 for an arithmetic (X, ω) with one or more cone points. For given rational slope α and given cone point $p \in X$: how many leaves $\mathcal{L} \in \mathcal{F}_\alpha(X^2)$ contain p and how many squares are needed to tile \mathcal{L} ? What are the geometric and dynamic properties of \mathcal{L} ? How are these properties related to the properties of the marked surfaces parametrized by \mathcal{L} . Ask the same question for a modular fiber \mathcal{F} covering X^2 , i.e. \mathcal{F} parametrizes branched covers of X . Can we say something about connectedness of \mathcal{F} ?

5. INVARIANT SUBSPACES OF \mathbb{T}^4

If $(X, \omega) = (\mathbb{R}^2/\mathbb{Z}^2, dz) =: \mathbb{T}^2$ there are infinitely many $\mathrm{SL}_2(\mathbb{Z})$ invariant, complex subspaces in every dimension ranging from 0 to n in \mathbb{T}^{2n} . The description of these subspaces was (implicitly) given in [S1]. From the geometric point of view all embedded unions of complex-linear tori which define subgroups are $\mathrm{SL}_2(\mathbb{Z})$ invariant subspaces of \mathbb{T}^{2n} . We do not describe the higher dimensional case, i.e. subspaces of X^n at this place, because it's a straightforward generalization of the 2 complex dimensional case.

Definition 18. We define a $\mathrm{SL}_2(\mathbb{Z})$ -action on \mathbb{T}^4 , by

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Z}) \times \mathbb{T}^4 & \xrightarrow{\odot} \mathbb{T}^4 \cong \mathbb{T}^2 \times \mathbb{T}^2 \\ (A, z) = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (z_h, z_v) \right) & \longmapsto A \odot (z_h, z_v) := (az_h + bz_v, cz_h + dz_v) \end{aligned} \quad (21)$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Because the \odot -action is complex-linear, it *commutes* with the diagonal-action of $\mathrm{SL}_2(\mathbb{Z})$ on \mathbb{T}^4 . Indeed with $z_h := x_h + iy_h$ and $z_v := x_v + iy_v$ one easily verifies

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix} \odot \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot_d \begin{pmatrix} x_h + iy_h \\ x_v + iy_v \end{pmatrix} \right] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \odot_d \left[\begin{pmatrix} e & f \\ g & h \end{pmatrix} \odot \begin{pmatrix} x_h + iy_h \\ x_v + iy_v \end{pmatrix} \right] \quad (22)$$

We call an $\mathrm{SL}_2(\mathbb{Z})$ invariant subspace $(S, \omega_S) \subset \mathbb{T}^4$ *simple*, if there is a $z \in S$ with $S = \overline{\mathrm{SL}_2(\mathbb{Z}) \cdot z}$, i.e. if S is the closure of the $\mathrm{SL}_2(\mathbb{Z})$ orbit of a single point.

Ratner's Theorem implies that the that $\mathrm{SL}_2(\mathbb{Z})$ -invariant subspaces of the Lie-group \mathbb{T}^4 are Lie-subgroups. Thus the classification of $\mathrm{SL}_2(\mathbb{Z})$ -invariant subspaces is well known and easy to achieve. The purpose of this section is to characterize invariant subspaces using the slope defined earlier. To state the theorem it is useful to write

$$\mathcal{O}_n := \left\{ \left[\frac{a}{n}, \frac{b}{n} \right] : a, b, n \in \mathbb{Z}^2, (a, b, n) = 1 \right\} = \mathrm{SL}_2(\mathbb{Z}) \cdot \left[\frac{1}{n}, 0 \right] \subset \mathbb{T}^2.$$

Theorem 19. The following statements are equivalent

1. $S \subset \mathbb{T}^4$ is a 2-dimensional, simple $\mathrm{SL}_2(\mathbb{Z})$ invariant subspace
2. Given $S \subset \mathbb{T}^4$, there is an $A \in \mathrm{SL}_2(\mathbb{Z})$ and an $n \in \mathbb{N}$, such that $S_h := A \odot S$ is horizontal and $S_h \cap \mathbb{T}_v^2 = \mathcal{O}_n$.
3. There is a nontrivial, $\mathrm{SL}_2(\mathbb{Z})$ -equivariant homomorphism of complex Lie groups $\pi : \mathbb{T}^4 \cong \mathbb{T}^2 \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$, which does not factor over an isogeny and an $n \in \mathbb{N}$, such that $S = \pi^{-1}(\mathcal{O}_n)$.

4. S consists of solutions of $az_h + bz_v \equiv 0 \pmod{\mathbb{Z}^2}$ with $a, b \in \mathbb{Z}$ and $(a, b) = n$ which do not solve any equation with coefficients $(a, b) = m$ for $m|n$.

Proof. From Theorem 3 we know that all invariant subspaces are defined by a real, linear equation. Hence all invariant subspaces are contained in the foliations $\mathcal{F}_\alpha(\mathbb{T}^2 \times \mathbb{T}^2)$, $\alpha \in \mathbb{R} \cup \{\infty\}$. It is well known that leaves of $\mathcal{F}_\alpha(\mathbb{T}^2 \times \mathbb{T}^2)$ with irrational slope are dense in \mathbb{T}^4 . Thus any closed, invariant $S \subset \mathcal{F}_\alpha(\mathbb{T}^2 \times \mathbb{T}^2)$ is contained in $\mathcal{F}_\alpha(\mathbb{T}^2 \times \mathbb{T}^2)$, with $\alpha \in \mathbb{Q} \cup \{\infty\}$. But for rational α there is an $A \in \mathrm{SL}_2(\mathbb{Z})$, such that $A \odot \mathcal{F}_\alpha(\mathbb{T}^2 \times \mathbb{T}^2) = \mathcal{F}_h(\mathbb{T}^2 \times \mathbb{T}^2)$. Since the real-action commutes with the diagonal action, we know $S_h := A \odot S \in \mathcal{F}_h(\mathbb{T}^2 \times \mathbb{T}^2)$ is $\mathrm{SL}_2(\mathbb{Z})$ invariant and for the same reason S_h is simple, if S is. The $\mathrm{SL}_2(\mathbb{Z})$ -orbit classification on \mathbb{T}^2 then implies there is an $n \in \mathbb{N}$ such that $S_h \cap \mathbb{T}_h^2 = \mathcal{O}_n$. Obviously (2) implies (1), showing the equivalence of statement (1) and (2). Note, that each component of S_h is a torus \mathbb{T}^2 , which is indeed the orbit closure of a single (irrational) point under the action of $\mathrm{SL}_2(\mathbb{Z})$.

(2) \Leftrightarrow (3) Take $A \in \mathrm{SL}_2(\mathbb{Z})$ making S horizontal and look at the map

$$\psi : \begin{array}{ccc} \mathbb{T}^4 & \longrightarrow & \mathbb{T}^2 \times \mathbb{T}^2 & \longrightarrow & \mathbb{T}^2 \\ z & \longmapsto & (z_h, z_v) := A \odot z & \longmapsto & z_v \end{array} . \quad (23)$$

Since both maps in the composition are linear and commute with the (diagonal) action of $\mathrm{SL}_2(\mathbb{Z})$, π is a $\mathrm{SL}_2(\mathbb{Z})$ -equivariant homomorphism of the complex Lie groups $\mathbb{T}^2 \times \mathbb{T}^2$ and \mathbb{T}^2 . Now π cannot factor over an isogeny

$$\psi : \mathbb{T}^2 \rightarrow \mathbb{T}^2, \quad [z] \mapsto [az] \quad \text{where } a \in \mathbb{Z},$$

because $\det A = 1$. The image of π is by construction \mathcal{O}_n for some $n \in \mathbb{N}$. On the other hand every complex Lie group homomorphism $\pi : \mathbb{T}^2 \times \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is given by

$$(z_h, z_v) \mapsto cz_h + dz_v \quad \text{with } c, d \in \mathbb{Z}.$$

If π does not factor over an isogeny we must have $(c, d) = 1$ and we can extend π to a linear map

$$A : \mathbb{T}^2 \times \mathbb{T}^2 \rightarrow \mathbb{T}_h^2 \times \mathbb{T}_v^2, \quad A \in \mathrm{SL}_2(\mathbb{Z})$$

where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now the image of $S = \pi^{-1}(\mathcal{O}_n)$ under A is horizontal and simple because $\mathcal{O}_n \subset \mathbb{T}^2$ is.

For (2) \Rightarrow (4) we note that any $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ making S horizontal provides us with \mathcal{O}_n equations characterizing S

$$cz_h + dz_v \equiv \left[\frac{p}{n}, \frac{q}{n} \right] \pmod{\mathbb{Z}^2}, \quad \text{where } \left[\frac{p}{n}, \frac{q}{n} \right] \in \mathcal{O}_n. \quad (24)$$

These can be expressed by the equation

$$ncz_h + ndz_v \equiv 0 \pmod{\mathbb{Z}^2}.$$

Now the statement follows because $(c, d) = 1$. For the reverse implication we easily see that given an equation $ncz_h + ndz_v \equiv 0 \pmod{\mathbb{Z}^2}$ with $(c, d) = 1$, the set of solutions which solves no equation of the shape $mcz_h + mdz_v \equiv 0 \pmod{\mathbb{Z}^2}$ for any $m|n$ solves precisely the $|\mathcal{O}_n|$ equations (24). Using c and d as second row in a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ in turn defines a map which makes the set of solutions of the given equation horizontal and has the desired intersection property. \square

Proposition 20. *Let Y be a subset of \mathbb{T}^4 which is closed, invariant under the diagonal action of $\mathrm{SL}_2(\mathbb{Z})$ and contains infinitely many invariant two dimensional torii then Y is equal to \mathbb{T}^4 .*

We first prove an arithmetical lemma:

Lemma 21. *Given any $\varepsilon > 0$, the orbit*

$$\mathcal{O}_n = \mathrm{SL}_2(\mathbb{Z}) \cdot \left[\frac{1}{n}, 0 \right] \subset \mathbb{T}^2$$

is ε dense in \mathbb{T}^2 for n large enough.

Proof of Lemma 21. The set \mathcal{O}_n contains the set

$$\left\{ \left[\frac{a}{n}, \frac{b}{n} \right] : a, b, n \in \mathbb{Z}, \gcd(a, n) = 1 \text{ and } \gcd(b, n) = 1 \right\}.$$

Therefore, it is enough to prove that the set $\left\{ \frac{a}{n} : a, n \in \mathbb{Z}, (a, n) = 1 \right\}$ is ε dense in \mathbb{T}^1 for n large enough.

The Jacobsthal's function $J(n)$ is the largest gap between consecutive integers relatively prime to n and less than n . We have to prove that $J(n)/n$ tends to 0 as n tends to infinity to complete the proof of lemma 21. In fact, a much more precise result does exist. By Iwaniec's result (see [I]) there is a constant K such that $J(n) \leq K(\ln n)^2$. This ends the proof of lemma 21. \square

Proof of Proposition 20. We use the classification of the orbit closures obtained in part 4) of Theorem 19: for the r th torus S_r we denote the coefficients of the equation of the torus by a_r, b_r, n_r . If the coefficients are uniformly bounded (in r) then there are only a finite number of tori, thus at least one of the coefficients is not bounded. Given $\varepsilon > 0$, if n_r is sufficiently large then the orbit of the torus S_r (by Theorem 19 part 2) is ε dense since it consists of $\varphi(n)\psi(n)$ parallel tori with transverse spacing at most ε by Lemma 21.

On the other hand, if n_r is bounded then the slope sequence a_r/b_r has a limit point (maybe 0 or infinity). If this limit “slope” is irrational then clearly $Y = \mathbb{T}^4$. In the rational case we also have $Y = \mathbb{T}^4$, since the area of the S_r approach infinity with transverse spacing approaching 0. \square

6. PROOF OF THEOREM 5 AND THEOREM 7

With the previous observations we are ready to prove Theorem 5:

Proof of Theorem 5 for X^2 . If X is arithmetic then the invariant subspaces are lifts of the invariant subspaces of \mathbb{T}^4 . Since there are infinitely many invariant subspace of \mathbb{T}^4 (see Theorem 19) there are infinitely many for X .

Let S be a 2-dimensional, $\text{Aff}^+(X, \omega)$ invariant submanifold of X^2 with nontrivial slope. Then the two projections

$$X_h \xleftarrow{\text{pr}_h} S \xrightarrow{\text{pr}_v} X_v \quad (25)$$

are surjective and every point $p = (p_h, p_v) \in S$ with one periodic coordinate is already periodic, by Proposition (13). If (X, ω) is not arithmetic, it contains only finitely many periodic points (see [GHS]). Thus in the non-arithmetic case, we can look at the surface (X, p_1, \dots, p_n) , marked in all the periodic points (some of them might be cone points). Note that by assumption the set of periodic points is not empty, because $g(X) \geq 2$ for non-arithmetic (X, ω) .

Any completely periodic foliation $\mathcal{F}_\theta(X)$ on (X, p_1, \dots, p_n) contains a nonempty *spine*, i.e. the weighted graph of saddle connections contained in $\mathcal{F}_\theta(X)$, the weights being the lengths of the saddle connections. The spine is nonempty by the existence of cone points. With respect to the two projections above any invariant subspace S defines a *similarity* of spines from X to itself, i.e. a bijective map of the spines of $\mathcal{F}_\theta(X_h)$ and $\mathcal{F}_\theta(X_v)$ stretching the length of each saddle connection by a factor $|\alpha|$, $\alpha \in \mathbb{R}^*$ the slope of S .

Without restrictions we can assume $\alpha \in \mathbb{R}^*$ satisfies $|\alpha| \geq 1$, otherwise we change the roles of pr_h and pr_v . Now take the longest saddle connection in the horizontal spine of X . If $|\alpha| > 1$ this saddle connection would be mapped to a saddle connection stretched by a factor $|\alpha|$, contained in the horizontal spine of X . Contradiction, thus we must have $\alpha = \pm 1$.

For slope ± 1 subspaces $S \subset X^2$ we first note that S intersects the fiber $\{p_h\} \times X_v \subset X_h \times X_v$, where $p_h \in X$ is any cone point. To see this

take a point $(z_h, z_v) \in S \subset X_h \times X_v$ and connect z_h with $p_h \in X$ using a path in $X = X_h$, this path lifts to S . In other words, the two coordinate projections of $S \rightarrow X_i$, $i = h, v$ are surjective. By Proposition 13 the intersection $S \cap \{p_h\} \times X_v$ consists of periodic points, since $p_h \in X$ is periodic. Thus the finiteness of the number of subspaces follows by the finiteness of periodic points and finiteness of orders of cone-points on X by lemma 16. Note that any neighborhood $U \subset S \subset X^2$ (S invariant) of a periodic point determines S by ergodicity of the $\mathrm{SL}(X, \omega)$ action on S . \square

Proof of Theorem 3 and 5 for X^n , $n \geq 3$. We prove the claim by induction over the complex dimension n of X^n . We already have the result for $n = 1$ (finitely many periodic points) and for $n = 2$. The claim is clear, if X is arithmetic, thus we assume X is not arithmetic. In this case we can also assume all periodic points of X are fix points. This does not change the invariant subspaces $S \subset X^n$, since the group $\Gamma \subset \mathrm{SL}(X, \omega)$ stabilizing all periodic points of X is a finite index subgroup of $\mathrm{SL}(X, \omega)$. Without restrictions we assume $\mathrm{SL}(X, \omega)$ stabilizes all periodic points. Denote by

$$pr_i : X^n \rightarrow X_i^{n-1} := X_1 \times \dots \times X_{i-1} \times X_{i+1} \times \dots \times X_n$$

the projections to the n hyperplanes X_i^{n-1} . Each periodic(= fix) point $p \in X$ defines an embedding

$$e_{i,p} : X_i^{n-1} \rightarrow X_1 \times \dots \times X_{i-1} \times \{p\} \times X_{i+1} \times \dots \times X_n \subset X^n$$

of each hyperplane X_i^{n-1} with the property

$$pr_i \circ e_{i,p} = \mathrm{id}_{X_i^{n-1}}, \text{ for all } i = 1, \dots, n.$$

For any *invariant* hypersurface $S \subset X^n$ and any $\mathrm{SL}(X, \omega)$ -fix point $p \in X$ we have

$$\dim_{\mathbb{C}}(e_{i,p}(X_i^{n-1}) \cap S) = \begin{cases} n-1 \\ n-2 \\ 0 \end{cases} . \quad (26)$$

In the first case $\mathrm{SL}(X, \omega) \cdot e_{i,p}(X_i^{n-1}) \cong S$, while in the last case $e_{i,q}(X_i^{n-1}) \cong S$ for an $\mathrm{SL}(X, \omega)$ -fix point $q \neq p$. In the remaining case S is characterized by n intersections, which are invariant subspaces of lower dimension. In fact, choosing an $\mathrm{SL}(X, \omega)$ -fix point $p \in X$ and considering the n hyperplanes $X_i^{n-1} := e_{i,p}(X_i^{n-1})$ shows that S is defined by the n intersections $S \cap X_i^{n-1}$.

If $\dim_{\mathbb{C}}(S) \leq n-2$ we can describe S as an intersection of invariant

surfaces of dimension less than $n - 1$:

$$S = \bigcap_i \text{pr}_i^{-1} \circ \text{pr}_i(S). \quad (27)$$

The statement follows now by induction over the dimension n . \square

Proposition 22. *Given a reduced lattice surface (X, ω) , then the diagonal (D_+, ω_+) and the off-diagonal (if it exists) (D_-, ω_-) , $(D_\pm \subset X^2)$ are translation surfaces isomorphic to (X, ω) .*

Proof. For the diagonal $\omega_\pm = \sqrt{2} \cdot \omega$ is induced by the bijective translation

$$\begin{aligned} D_+ &\longrightarrow X \\ (x, x) &\longmapsto x \end{aligned} \quad (28)$$

while for the offdiagonal $w_- = \sqrt{2}\omega$ is induced by

$$\begin{aligned} D_- &\longrightarrow D_+ \\ (x, \phi(x)) &\longmapsto (x, x) \end{aligned} \quad (29)$$

where $\phi \in \text{Aff}^+(X, \omega)$ is the unique map, such that $D\phi = -\text{id}$. \square

Proof of Theorem 4. All invariant subspaces $S \subset X^n$ of complex dimension 1 are real-linear and $\text{Aff}^+(X, \omega)$ acts as a group of real-affine homeomorphisms with respect to the given complex-linear structure. We obtain the claim after noticing that each connected component of S has a lattice stabilizer in $\text{Aff}^+(X, \omega)$, in particular this stabilizer is $\text{SL}(X, \omega)$ if S is connected. Note that as a Veech surface a priori S could have an even bigger Veech- and affine group.

We explicitly define the differential ω_S on S using local parametrizations of S which are induced by natural charts of (X, ω) .

By real linearity of S there are vectors $(a_1, \dots, a_n) \in \mathbb{R}^n$, $(t_1, \dots, t_n) \in \mathbb{C}^n$ and n local translation maps $S \rightarrow X$, such that

$$\phi : \begin{cases} S \hookrightarrow S \subset X^n \\ z \longmapsto (a_1 z + t_1, a_2 z + t_2, \dots, a_n z + t_n) \end{cases} \quad (30)$$

defines a local homeomorphism. Now the 1-form on $S \subset X^n$ defined by

$$\frac{1}{\|a\|^2} \sum_{i=1}^n a_i \omega_i|_{TS}, \quad \text{with } \|a\|^2 := \sum_{i=1}^n a_i^2$$

implies that locally

$$\phi^* \omega_S = \phi^* \left(\frac{1}{\|a\|^2} \sum_{i=1}^n a_i \omega_i|_{TS} \right) = \frac{1}{\|a\|^2} \sum_{i=1}^n a_i^2 dz = dz \quad (31)$$

showing that (30) defines a translation map with respect to natural charts. The diagonal action of $\text{Aff}^+(X, \omega)$ on X^n induces an action of $\text{SL}(x, \omega)$ on ω_S , given by

$$A \cdot \omega_S := \frac{1}{\|a\|^2} \sum_{i=1}^n a_i A \cdot \omega_i|_{TS} = A \cdot \frac{1}{\|a\|^2} \sum_{i=1}^n a_i \omega_i|_{TS} = A \cdot \omega_S$$

The last identity is caused by real linearity of $A \in \text{SL}(x, \omega)$. But this is the claim, i.e. the diagonal action induces the standard action of $\text{Aff}^+(X, \omega)$ on (S, ω_S) given by

$$A \cdot \omega_S := (1, i) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \text{Re}(\omega_S) \\ \text{Im}(\omega_S) \end{pmatrix}, \quad \text{if} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (32)$$

□

Lemma 23. *Take a lattice surface (X, ω) . Assume there is a periodic direction θ , such that all absolute periods represented by leaves $\mathcal{L} \in \mathcal{F}_\theta(X)$ fulfill $|\mathcal{L}| \in \alpha \cdot \mathbb{Q}$, $\alpha \in \mathbb{R}_+$. Then (X, ω) is arithmetic.*

Proof. Without restrictions we can assume $\alpha = 1$, the periodic direction θ is horizontal and the vertical foliation of (X, ω) is periodic too. Take the horizontal cylinder decomposition of (X, ω) , the i -th cylinder having height h_i and width w_i . Since (X, ω) is a Veech surface we have $w_i/h_i \cdot h_j/w_j \in \mathbb{Q}$ and thus $h_i/h_j \in \mathbb{Q}$ for all pairs of cylinder(-heights), because all w_i are rational. This implies

$$h_1, \dots, h_n \in \alpha_h \cdot \mathbb{Q}$$

and again we can assume $\alpha_h = 1$. Since all the width and the heights of cylinders, including the absolute horizontal periods, which might represent cylinders of height 0, are rational, they define a lattice in \mathbb{R}^2 . This in turn defines a translation map $\pi : (X, \omega) \rightarrow (\mathbb{R}^2 / \text{Per}(\omega), dz)$, proving the Lemma. Note that by assumption $\text{Per}(\omega) \subset \mathbb{Z}^2$.

To represent π (see also [Z2]), choose a cone point $p \in X$ and any other point $z \in X$. Then any path connecting p with z is isotopic (in $X - Z(\omega)$) to a path consisting entirely of horizontal and vertical line segments. Now take the holonomy of the straightened path modulo $\text{Per}(\omega)$ and note that any other path from p to z gives the same vector modulo $\text{Per}(\omega)$. □

Proof of Theorem 7. First note, that for primitive and by definition for reduced lattice surfaces (X, ω) taking derivatives induces an isomorphism $\text{SL}(X, \omega) \cong \text{Aff}^+(X, \omega)$. Thus we can assume $\text{SL}(X, \omega)$ acts on X^2 .

By Theorem 19 the statements hold for arithmetic surfaces. Thus we

might assume (X, ω) is not arithmetic and all the (finitely many) periodic points of (X, ω) are marked.

Assume $S \subset X^2$ is a $\text{Aff}^+(X, \omega)$ invariant subspace of slope ± 1 . Because we mark all periodic points of X , S induces a length preserving bijection of saddle connections for each completely periodic direction. Now take a cone point $p = (p_h, p_v) \in S \subset X_h \times X_v$, such that (without restrictions) p_h is the (one and only) cone point of $X = X_h$. We have two cases:

p_v is not a cone point, but periodic. Then for any given periodic direction θ , we consider a (chain of) saddle connection(s) $\mathcal{L}_{p_v} \in \mathcal{F}_\theta(X)$ connecting p_v with itself, containing p_v only once. Again there are two cases:

$\mathcal{L}_{p_v} \in \mathcal{F}_\theta(X)$ is a regular leaf. Then S sets up a correspondence of $\mathcal{L}_\theta(p_v)$ and all o_{p_h} leaves $\mathcal{L}_{p_h, i} \in \mathcal{F}_\theta(X)$ $i = 1, \dots, o_{p_h}$ starting and ending at p_h must have length

$$|\mathcal{L}_{p_v}| = |\mathcal{L}_{p_h, i}| \quad \text{for all } i = 1, \dots, o_{p_h}$$

The other case is, if $p_h \in \mathcal{L}_{p_v}$ $p_h = p$ the cone point of (X, ω) . Again we require that \mathcal{L}_{p_v} meets p_h just one time. Then $\mathcal{L}_{p_v} = \mathcal{L}_{p_h, i}$ for one i , say $i = 1$ and via the correspondence defined by S we find again

$$|\mathcal{L}_{p_h, 1}| = |\mathcal{L}_{p_h, i}| \quad \text{for all } i = 1, \dots, o_{p_h}.$$

This finishes the first case.

In the second case $(p_h, p_v) = (p, p)$ is the cone point of (X, ω) in both coordinates. Again we denote all o_{p_h} singular leaves connecting $p = p_h$ with itself in a completely periodic direction θ by $\mathcal{L}_{p_h, i}$, $i = 1, \dots, o_{p_h}$. We know that locally there are o_{p_h} 2-dimensional leaves in $\mathcal{F}_{\pm 1}(X^2)$ containing (p, p) , labeled by which leave $\mathcal{L}_{p_h, i}$ we identify with $\mathcal{L}_{p_h, 1}$. If o_{p_h} is prime every nontrivial combination will again give

$$|\mathcal{L}_{p_h, 1}| = |\mathcal{L}_{p_h, i}| \quad \text{for all } i = 1, \dots, o_{p_h}$$

by cyclicity of the identifications. The trivial identification, i.e. $\mathcal{L}_{p_h, 1}$ with $\mathcal{L}_{p_v, 1}$, is contained in D_\pm . Both cases imply with Lemma 23: either (X, ω) is arithmetic in contradiction to our assumption, or $S = D_\pm$. \square

Note: we prove more, than stated in Theorem 7, in fact we can say

Corollary 24. *Let (X, ω) be a reduced Veech surface with exactly one cone-point $p \in X$, then all 2-dimensional $\text{SL}(X, \omega)$ -invariant subspaces $S \subset X^2$ with non-trivial slope contain the cone point $(p, p) \in X^2$.*

The Corollary implies that X^2 , (X, ω) a reduced, non-arithmetic lattice surface with exactly one cone point, cannot contain more than $o = o_p$ connected, invariant surfaces of slope ± 1 . It follows from Proposition 13 and Lemma 16 that for non-arithmetic Veech surfaces (X, ω) the number of compact leaves (with slope ± 1) is bounded by

$$\begin{aligned} \#\{S \subset X^2 : \dim_{\mathbb{C}}(S) = 1, \text{ slope}(S) = \pm 1\} &\leq \\ &\leq \frac{\sum_{\substack{p, q \in X \text{ periodic}}} o_p \cdot o_q}{\sum_{\substack{p \in X \text{ periodic}}} o_p} = \sum_{\substack{p \in X \text{ periodic}}} o_p, \end{aligned}$$

which is the *total order* of periodic points on (X, ω) . To make this bound effective one can use Mller's result [M1] for the number of periodic points on non-arithmetic Veech surfaces of genus 2 with one cone point. We obtain the upper bound 8 for the number of (off-) diagonals for genus 2 surfaces with one cone point.

Another immediate consequence is

Corollary 25. *All primitive Veech surfaces (X, ω) of genus g , with exactly one cone point, such that $2g - 1$ is prime have only D_+ and D_- as 2-dimensional, $\text{SL}(X, \omega)$ -invariant subspaces of X^2 . D_- exists if and only if $\text{SL}(X, \omega)$ admits the element $-\text{id}$. In particular for all primitive Veech surfaces (X, ω) with one cone point in genus 2, 3 and 4 the only invariant surfaces are D_- and D_+ .*

Proof. We have $o_p = 2g - 1$, if there is only one cone point p on (X, ω) and $o_p = 3, 5, 7$ for $g = 2, 3, 4$. \square

Note that all known primitive Veech surfaces in genus 2, 3 and 4 possess an involution, i.e. $-\text{id} \in \text{SL}(X, \omega)$.

7. THE ILLUMINATION PROBLEM — PRELATTICE SURFACES

A discrete group $\Gamma \subset \text{SL}_2(\mathbb{R})$ called *prelattice*, if it contains noncommuting parabolic elements. A translation surface (X, ω) is a *prelattice surface* if $\text{SL}(X, \omega)$ is a prelattice group. By acting by $\text{SL}_2(\mathbb{R})$ we see that if a surface is prelattice then we can always assume that the two parabolic elements are orthogonal, one horizontal one vertical.² Thus we can decompose the surface into vertical cylinders and also into horizontal cylinders. Taking the intersection of the cylinders leads to a decomposition into rectangles with vertical and horizontal sides. For

²We make this assumption for simplicity of notation, it affects none of our results.

$p \in X$ let $\mathcal{C}_v(p)$ be the vertical cylinder, $\mathcal{C}_h(p)$ the horizontal cylinder and $\mathcal{R}(p)$ the rectangle containing p .

Let

$$T_h := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T_v := \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$$

be the horizontal and vertical parabolic elements in $\text{SL}(X, \omega)$. Here a and b depend on the prelattice surface (X, ω) . Suppose \mathcal{C} is a vertical cylinder with height h and width w and with origin the bottom left corner. Then the action of T_v on \mathcal{C} is given by a power of the Dehn twist:

$$T_v \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + bx \ (h) \end{pmatrix}. \quad (33)$$

Here (h) means modulo h . Similarly for a horizontal cylinder \mathcal{C} the action of T_h is given by

$$T_h \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + ay \ (h) \\ y \end{pmatrix}. \quad (34)$$

Warning, in the standard notation which we follow here the height of a horizontal cylinder is noted w and the width is noted h .

If $\text{SL}(X, \omega)$ is a lattice, then (X, ω) is a *Veech surface*. Veech has shown that the direction of any saddle connection on a Veech surface is a parabolic direction [V1].

Proposition 26. *The set of $\{(p, q) : p \text{ illuminates } q\}$ is invariant under the diagonal action of $\text{Aff}^+(X, \omega)$, i.e. if p illuminates q then $\phi(p)$ illuminates $\phi(q)$ for all $\phi \in \text{Aff}^+(X, \omega)$.*

This follows from the fact that ϕ takes a geodesic to a geodesic.

A key observation is the following simple fact.

Proposition 27. *Fix a vertical (or horizontal) cylinder $\mathcal{C} \subset X$. Each point $p \in \mathcal{C}$ illuminates every $q \in \mathcal{R}(p)$.*

This follows since a rectangle is convex. Theorem 1 follows immediately from Propositions 26 and 27 combined with the following more general theorem.

Theorem 28. *Suppose that (X, ω) is a prelattice surface and fix $p \in X$. Then except for an at most countable exceptional set of $q \in X$ there exists a $\phi := \phi(p, q) \in \text{Aff}^+(X, \omega)$ such that $\phi(p)$ and $\phi(q)$ are both contained in the rectangle $\mathcal{R}(p)$.*

Part of the proof of Theorem 28 is based on the arithmetic analysis of rotations of the torus. We consider the two torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ and

for $(c, d) \in \mathbb{T}^2$ let $R_{\varphi, \vartheta}^n(c, d)$ be the rotation by (φ, ϑ) . We need the following quantitative version of Kronecker's theorem.

Lemma 29. *Fix $(c, e) \in \mathbb{S}^1 \times \mathbb{S}^1 \cong \mathbb{T}^2$, $\varphi \in \mathbb{S}^1$ and $\varepsilon > 0$. Then for all, but finitely many $\vartheta \in \mathbb{S}^1$ and for all $d \in \mathbb{S}^1$ there exists an n , such that $R_{\varphi, \vartheta}^n(c, d) \in (c - \varepsilon, c + \varepsilon) \times (e - \varepsilon, e + \varepsilon)$.*

Remark: the cardinality of the exceptional set depends on φ and ε .

Proof. Consider a $\vartheta \in \mathbb{S}^1$. There are three cases according to the dimension Δ of the vector space over \mathbb{Q} generated by $(1, \varphi, \vartheta)$.

Suppose this dimension is 3. The classical Kronecker theorem asserts that for all d , the orbit of (c, d) by $R_{\varphi, \vartheta}^n$ is dense in \mathbb{T}^2 ([KH], page 29) and in this case the exception set of ϑ is empty.

Next suppose the dimension satisfies $\Delta = 1$. Thus φ and ϑ are rational numbers, we note them in reduced form as $\varphi = p_1/q_1$ and $\vartheta = p_2/q_2$. We emphasize that the number φ is fixed. For any n the point $R_{\varphi, \vartheta}^{nq_1}(c, d)$ has first coordinate c . Thus we think of $R_{\varphi, \vartheta}^{nq_1}(c, d)$ as a rotation of the circle $\{c\} \times \mathbb{S}^1$. Thus if θ is rational with denominator q_2 greater than $1/\varepsilon$, then orbit $R_{\varphi, \vartheta}^{nq_1}$ of any point (c, d) is ε -dense in the circle $\{c\} \times \mathbb{S}^1$ and the lemma holds with the exceptional set of ϑ (a subset of) those with denominator less than $1/\varepsilon$.

Finally consider the case when $\Delta = 2$. Thus there exists $A, B, C \in \mathbb{Z}$ with $\gcd(A, B, C) = 1$ such that $A\varphi + B\vartheta = C$. Let $D := \gcd(A, B)$, $A' := A/D$ and $B' := B/D$. Then the above equation becomes $A'\varphi + B'\vartheta = C/D$.

By homogeneity it suffices to prove the lemma for $c = d = 0$. The orbit of $(0, 0)$ is contained in the torus-loops L_k : $A'x + B'y = k/D \pmod{1}$ with $k \in \{0, 1, \dots, D-1\}$.

Consider $R_{\varphi, \vartheta}^{nk}(0, 0)$. Since $nD(A'\varphi + B'\vartheta) = nD(C/D) = 0 \pmod{1}$ these points are contained in the torus-loop L_0 . As in the case $\Delta = 3$ the action of $R_{\varphi, \vartheta}^{nD}$ on the circle L_0 is an irrational rotation, therefore the orbit of $(0, 0)$ is dense in L_0 .

Suppose that $R_{\varphi, \vartheta}^n(0, 0) \in L_k$. Then we claim that $R_{\varphi, \vartheta}^{n+1}(0, 0) \in L_{k'}$ with $k' := (k + C) \pmod{D}$. To see this note that $n(A'\varphi + B'\vartheta) = \frac{k}{D}$ implies that

$$(n+1)(A'\varphi + B'\vartheta) = \frac{k}{D} + (A'\varphi + B'\vartheta) = \frac{(k + C) \pmod{D}}{D} = \frac{k'}{D}.$$

Since $\gcd(C, D) = 1$ the map $R_{\varphi, \vartheta}^n$ permutes the circles L_k cyclically. Thus the orbit of $(0, 0)$ is dense in $\mathbb{L} := \cup_{k=0}^{D-1} L_k$.

We claim that if A, B, C are sufficiently large that then \mathbb{L} is ε -dense in \mathbb{T}^2 . There are five subcases.

i) Suppose $B' = 0$. The circles L_k are vertical, and are given by the formula $x = k/|A|$. If $|A| > 1/\varepsilon$ then these circles are ε -dense.

ii) Suppose $A' = 0$. The circles L_k are horizontal, and the proof is similar to case i).

iii) If $B' \neq 0$ and $|A'| > 1/\varepsilon$ then each then each of the circles L_k is ε -dense and the lemma follows.

iv) The case $A' \neq 0$ and $|B'| > 1/\varepsilon$ is similar to case iii).

v) If D is large we have D parallel and equidistant circles, thus they are at least $1/D$ dense.

The only cases which do not fall into these five subcases are when $|A'|, |B'|$ and D are all smaller than $1/\varepsilon$, and thus $|A| = |A'D| < 1/\varepsilon^2$ and $|B| = |B'D| < 1/\varepsilon^2$. Furthermore $|C| \leq |A| + |B| < 2/\varepsilon^2$. Thus the exception set of θ is finite. \square

Proof of Theorem 28. Throughout the proof p and q stand for points in (X, ω) . We call a path connecting p to q a VH -path if it is the union of a finite number of vertical and horizontal segments. We will call a point in between a vertical and horizontal segment a turning point. We claim that in a prelattice surface every p and q can be connected by a finite length VH -path. We give an algorithm to produce simple VH -paths between two points. First each q in $\mathcal{C}_v(p)$ (resp. $\mathcal{C}_h(p)$) can be connect to p by a VH -path with one turning point. Next each $q' \in \mathcal{C}_v(q)$ or $\mathcal{C}_h(q)$ can be connected to p with a VH -path with two turning points, (see Figure 2) etc.

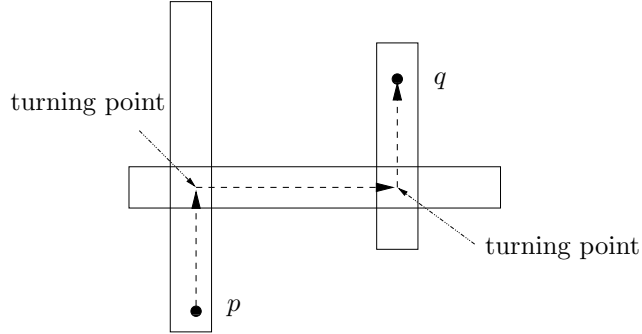


FIGURE 3. A VH -path.

Since the surface (X, ω) is connected the claim follows by exhaustion. From here on a VH -path will always be one of the simple paths produced by this algorithm. Fix p and q . Let r_i be the sequence of turning points of a VH -path connecting p to q and let $\mathcal{R}_i := \mathcal{R}(r_i)$.

Clearly the sequence \mathcal{R}_i depends only on $\mathcal{R}(p)$ and $\mathcal{R}(q)$ rather than on p and q . We call this sequence of rectangles the combinatorics of the rectangles $\mathcal{R}(p)$ and $\mathcal{R}(q)$.

Our strategy is as follows. We fix a point p and a rectangle $\mathcal{R} = \mathcal{R}(p)$. We are interested in the $q \in \mathcal{R}_N$ which are illuminated by p . We consider powers of T_h and T_v which keep p in $\mathcal{R}(p)$ and try to move q closer to p . More precisely fix the combinatorics of $\mathcal{R}(p)$ and $\mathcal{R}(q)$. By induction it suffices to show that all but countably many points in the rectangle \mathcal{R}_N , can be moved into rectangle \mathcal{R}_{N-1} by a power of T_h or T_v which keeps p in $\mathcal{R}(p)$.

To prove this we apply Lemma 29. Suppose that \mathcal{R}_N and \mathcal{R}_{N-1} are connected by a vertical path (i.e. they are in the same vertical cylinder \mathcal{C} of height h and width w). Remember that if \mathcal{C} is a vertical cylinder then on each vertical segment T_v acts as a rotation. We consider the vertical cylinders $\mathcal{C}_v(p) \times \mathcal{C}_v(q)$ and in this set we consider the product $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ consisting of the vertical closed curve through p times the vertical closed curve through q .³ The action of $T_v \times T_v$ is a translation on this torus. The angles of rotation are given by (33). Thus we can apply Lemma 29 with normalized coordinates $c := x/h$, $d := y/w$ and ε chosen such that $(c - \varepsilon, c + \varepsilon)$ corresponds (via the normalization) to the intersection the vertical closed curve through p and the rectangle $\mathcal{R}(p)$. Furthermore $(e - \varepsilon, e + \varepsilon)$ corresponds (via the normalization) to the intersection the horizontal closed curve through p and the rectangle $\mathcal{R}(q)$. We conclude that except for an exceptional set consisting of a finite union of vertical segments of \mathcal{R}_N all other points in \mathcal{R}_N can be moved to \mathcal{R}_{N-1} while the image of p stays in $\mathcal{R}(p)$. We call this union of vertical segments $\mathcal{B}_v(p)$.

Next we must apply now the horizontal twist. We apply Lemma 29 to the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ in $\mathcal{C}_h(p) \times \mathcal{C}_h(q)$ with ε chosen so that $(c - \varepsilon, c + \varepsilon) \subset \mathcal{R}(p)$ and $(e - \varepsilon, e + \varepsilon) \subset \mathcal{R}_N$. We conclude that except for an exceptional set consisting of a finite of horizontal segments of \mathcal{R}_N all other points in \mathcal{R}_N return to \mathcal{R}_N while the image of p stays in $\mathcal{R}(p)$. We call this union of horizontal segments $\mathcal{B}_h(p)$.

For each $k \geq 1$ consider the set $X_k := \{q \in \mathcal{R}_N : T_h^k(q) \in \mathcal{R}_N\}$. Thus $\cup_{k \geq 1} X_k = \mathcal{R}_N \setminus \mathcal{B}_h(p)$. The collection of points $\mathcal{B}_h(p) \cap \mathcal{B}_v(p)$ is finite. These points are exceptional points.

Applying the previous argument with the vertical twist yields that except of an exceptional set consisting of a finite union of vertical segments $\mathcal{B}_v(T_h^k(p))$ in $X_k \subset \mathcal{R}_N$ all other points in X_k can be moved

³Note that as q varies in $\mathcal{R}(q)$ these tori are isometric copies of each other, thus we can apply the abstract Lemma 29 as if they are all the same.

to \mathcal{R}_{N-1} while the image of $T_h^k(p)$ stays in $\mathcal{R}(p)$. Applying T_h^{-k} to $\mathcal{B}_v(T_h^k(p))$ yields a finite collection of segments of slope $-1/ak$ in \mathcal{R}_N . Since the sets $\mathcal{B}_v(q)$ and $T_h^{-k}(\mathcal{B}_v(T_h^k(p)))$ consist of a finite union of segments which intersect transversely the intersection $\mathcal{B}_v(q) \cap T_h^{-k}(\mathcal{B}_v(T_h^k(p)))$ is a finite set and the points in X_k which are not in this finite set can be moved to \mathcal{R}_{N-1} while the image of p stays in $\mathcal{R}(p)$. We conclude the theorem by taking the union over k . \square

8. ILLUMINATION ON VEECH SURFACES

Proof of Theorem 2. To prove Theorem 2 note that a Veech surface (X, ω) is necessarily a prelattice surface, thus we can apply Theorem 1.

Let us assume that X is a non arithmetic Veech surface. Given $p \in X$ we consider any convex open neighborhood \mathcal{O}_p of p and the *exceptional set*

$$\mathcal{E} := \{(z_h, z_v) \in X \times X : \text{Aff}^+(X, \omega) \circ_d (z_h, z_v) \notin \mathcal{O}_p \times \mathcal{O}_p\}.$$

The set \mathcal{E} is clearly closed and $\text{Aff}^+(X, \omega)$ invariant. Our results on invariant subspaces say, that \mathcal{E} is a finite union of orbit closures $\overline{\text{Aff}^+(X, \omega) \circ_d (z_1, z_2)}$, of complex dimension less than 2. For $p \in \mathcal{O}_p$, the set $\mathcal{E} \cap (\{p\} \times X)$ is algebraic. Additionally by Theorem 1, this set is at most countable, thus it is finite.

Let us assume now that X is an arithmetic surface.

We first suppose that p is a periodic point which means that both coordinates of p are rational. In this case, we refine the proof of theorem 1 to prove that p illumines every point q except a finite number. In the sequel, we use the notations of the proof of Theorem 1. Without loss of generality, we assume that p is fixed by horizontal and vertical parabolic elements T_h and T_v . Let $\mathcal{R}(q)$ be the rectangle defined by the horizontal and vertical cylinder decompositions of X containing q . We consider a VH-path from $\mathcal{R}(p)$ to $\mathcal{R}(q) = \mathcal{R}_N$. By induction, we show that all but *finitely many* points in the rectangle $\mathcal{R}(q)$ can be moved to \mathcal{R}_{N-1} . Suppose that \mathcal{R}_N and \mathcal{R}_{N-1} are in the same vertical cylinder. As in the proof of Theorem 1, applying T_v , we can move to \mathcal{R}_{N-1} every point of \mathcal{R}_N , except a finite number of segments denoted by $\mathcal{B}_v(p)$. The vertical coordinates of the elements of $\mathcal{B}_v(p)$ are rational numbers with small denominators. If q belongs to $\mathcal{B}_v(p)$, we apply the horizontal twist T_h . If the x coordinate of q is irrational, the orbit of T_h is dense in the horizontal circle containing q . If this coordinate is rational with a denominator large enough, it is ε dense. If ε is chosen small enough,

the orbit of q under T_h intersects $\mathcal{R}_n \setminus \mathcal{B}_v(p)$. Therefore, except for a finite number of points in $\mathcal{B}_v(p)$, this orbit intersects $\mathcal{R}_n \setminus \mathcal{B}_v(p)$. Thus, applying T_v , we move q to R_{N-1} .

If p is not a periodic point, we consider the exceptional set

$$\mathcal{E} := \{(z_h, z_v) \in X \times X : \text{Aff}^+(X, \omega) \circ_d (z_h, z_v) \notin \mathcal{O}_p \times \mathcal{O}_p\},$$

where \mathcal{O}_p is a convex open neighborhood of p . As p is not periodic, the fiber $\mathcal{E}_p = \mathcal{E} \cap (\{p\} \times X)$ does not contain any periodic point. Moreover, the intersection of \mathcal{E}_p with an invariant two dimensional orbit closure is finite. Therefore if the set \mathcal{E}_p is infinite, \mathcal{E} contains an infinite number of invariant two dimensional surface. Let π be the projection from X to \mathbb{T}^2 . The projection of \mathcal{E} is not $\text{SL}_2(\mathbb{Z})$ invariant. But, as $\text{SL}(X, \omega)$ is a finite index subgroup of $\text{SL}_2(\mathbb{Z})$, there exists a set Y , closed and $\text{SL}_2(\mathbb{Z})$ invariant, contained in \mathbb{T}^4 which is a *finite* union of images of $\pi(\mathcal{E})$. The set Y contains infinitely many two dimensional torii. By proposition 20, it is equal to \mathbb{T}^4 . This leads to a contradiction because Theorem 1 implies that the fiber over $\pi(p)$ is at most countable. Thus, the set \mathcal{E}_p is finite which completes the proof of theorem 2. \square

Proof of Theorem 8. We first prove the

Lemma 30. *Let X be a Veech surface then every point $p \in X$ which is non periodic is self-illuminating.*

Proof. Let p be a non periodic point. We take a periodic direction, say the horizontal one, and decompose X into maximal cylinders. Now we fix an open maximal cylinder \mathcal{C} , in this decomposition. Then the horizontal leaf through any $m \in \mathcal{C}$ is regular and thus m illuminates itself. If $p \in X$ is not periodic its $\text{Aff}^+(X, \omega)$ orbit is dense, hence there is a point, say $\psi(p)$ in the orbit of p which is in \mathcal{C} and therefore the horizontal leaf through $\psi(p)$ is regular. Thus p is self-illuminating. \square

By our results on invariant subspaces in arithmetic surfaces (X, ω) cannot be arithmetic. Without restrictions we assume that (X, ω) is reduced. Recall that a point $(p, q) \in X^2$, represents an illuminable configuration, i.e. there is a saddle connection between p and q , whenever its $\text{SL}(X, \omega)$ orbit intersects a convex set $\mathcal{O} \times \mathcal{O} \subset X^2$. In the sequel, we will assume that (p, q) is not a periodic point.

Now our classification says either

- $(p, q) \in X^2$ is generic and $D_+ \subset \overline{\text{SL}(X, \omega) \circ_d (p, q)}$, or

- (p, q) is part of an invariant surface (S, ω_S) .

All points in $X^2 \setminus D_+$ whose $\text{SL}(X, \omega)$ orbit closure contain the diagonal D_+ represent illuminable configurations. This is obvious if $p \neq q$ and a direct consequence of Lemma 30 if $p = q$.

If D_\pm are the only $\text{SL}(X, \omega)$ invariant surfaces of slope ± 1 , then all invariant surfaces in X^2 intersect D_+ . For horizontal or vertical subspaces this is clear. Denote the set of fix points under the (unique) affine involution by $\text{Fix}_\pm := D_- \cap D_+$.

Because Fix_\pm is an intersection of invariant surfaces, all its points are periodic points. Moreover the diagonal action of (X, ω) on X^2 restricted to invariant surfaces is ergodic too. This implies immediately that all nonperiodic points on parallel invariant surfaces are illuminable configurations. The only question remaining is what happens on D_- . The answer is:

- if all points in Fix_\pm are cone points, every nonperiodic point $(p, q) = (p, \phi(p)) \in D_-$ defines a non illuminable pairing
- if Fix_\pm contains one regular point all non periodic points on D_- define illuminable configurations on X .

The last statement follows again from ergodicity of the $\text{SL}(X, \omega)$ action restricted to D_- and from the

Lemma 31. *Let $\phi \in \text{Aff}^+(X, \omega)$ be the affine involution. The set Fix_\pm contains regular points if and only if there is $p \in X \setminus Z(\omega)$ and a saddle connection from p to $\phi(p)$.*

Proof. Given $p \in X \setminus Z(\omega)$ with $\phi(p) \neq p$, every saddle connection from p to $\phi(p)$ contains a fix point of ϕ , since $\phi(s) = -s$ ($-s$ is s with the opposite orientation). This follows because ϕ exchanges p and $\phi(p)$, while the angle of any outgoing or incoming saddle connection is turned by π .

The converse is obvious. \square

Finally note that if X is not reduced $\pi^{-1}(\text{Fix}_\pm) \subset X$ might only contain cone points, while Fix_\pm does not. Here $\pi : X \rightarrow X_{\text{red}}$ is the canonical cover. \square

Proof of Corollary 9. By Corollary 25 the Veech surfaces in view admit only the diagonal and off-diagonal as invariant subspaces. Since there are always regular Weierstrass points on genus 2 surfaces with one cone point, one only needs to consider pairs of periodic points. Moreover by Mllers result [M1] we know that the only periodic points on non-arithmetic Veech surfaces of genus 2 are the Weierstrass points. \square

General non-illumination or blocking configurations. Let (X, ω) a Veech surface with an involution $\phi \in \text{Aff}^+(X, \omega)$, then

$$D_\phi = \{(z, \phi(z)) \in X^2 : z \in X\}$$

is an invariant subset of slope -1 in X^2 . By the argument in the Proof of Theorem 8 Fix_ϕ provides a *blocking set* for $SC(z, \phi(z))$. The set Fix_ϕ consists of finitely many periodic points because $\text{Fix}_\phi = D_\phi \cap D_+$, hence we obtain a *finite blocking property* for all non-periodic pairs contained in D_ϕ . Because $D_\phi \cong X$ as translation surfaces, *generic* is equivalent to *not periodic* with respect to the action of the Veech-group.

Example. We again look at the staircase, but now we take only staircases with an even number $2n$ of steps. Denote by (Y_n, ω_n) the staircase with $2n$ steps, then we have the following:

- The absolute periods define a covering $\pi_n : (Y_n, \omega_n) \rightarrow (\mathbb{R}^2/2\mathbb{Z}^2, dz)$ branched over the points $\mathbb{Z}^2/2\mathbb{Z}^2$
- (Y_n, ω_n) has 4 cone points of order n , or equivalently ω_n has 4 zeros of order $n - 1$
- $g(Y_n) = 2n - 1$
- $\text{Aut}(Y_n, \omega_n) \cong \mathbb{Z}/4n\mathbb{Z}$
- $\text{Aut}_{\pi_n}(Y_n, \omega_n) := \{\phi \in \text{Aut}(Y_n, \omega_n) : \pi_n \circ \phi = \pi_n\} \cong \mathbb{Z}/n\mathbb{Z}$
- $\text{SL}(Y_n, \omega_n) \cong \Gamma_2$ unless $n = 1$, but then
- $(Y_1, \omega_1) \cong (\mathbb{T}^2, dz)$ and $\text{SL}(Y_1, \omega_1) \cong \text{SL}_2(\mathbb{Z})$

Lemma 32. For any point $z \in Y_n \setminus Z(\omega_n)$ there are n points, given by $\pi_n^{-1}(-\pi_n(z))$, which are not illuminable from z .

Proof. Assume there is a connecting geodesic, say s between z and $z_- \in \pi_n^{-1}(-\pi_n(z))$. Then $\pi_n(s)$ connects $\pi_n(z)$ and $\pi_n(z_-) = -\pi_n(z)$ and therefore contains a Weierstrass point of $\mathbb{R}^2/2\mathbb{Z}^2$. That means s intersects one of the preimages of Weierstrass points of $\mathbb{R}^2/2\mathbb{Z}^2$, but these are all cone points. \square

Consequently, there are surfaces (X, ω) such any point $z \in X$, which is not a cone point, admits several points not illuminable from z . In particular the total number of points which are not illuminable (from a given point) has no universal bound. Since the Veech-group of this example has not been used we drop it and present a more general class of surfaces with non-illumination pairs.

Non-Veech example with non-illumination points. Take any surface represented by an L shaped figure and construct the following double cover.

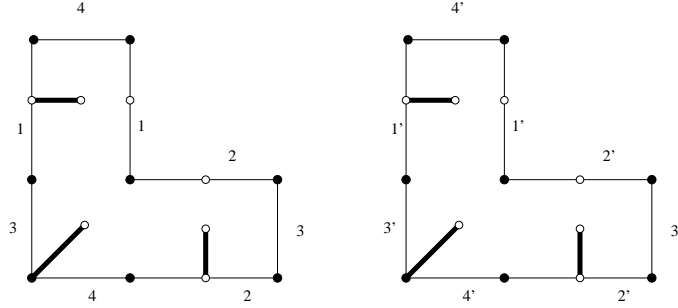


FIGURE 5. Double cover of L branched over Weierstrass points.

To obtain this cover, L_C , cut each copy of the L along the thick lines, and identify the two parallel slits so obtained cross wise. No matter what the width-height relations of L are, L_C is a double cover of L with cone points above all Weierstrass points of L . Recall that each L -shaped surface admits an involution.

By our previous arguments all straight geodesics connecting pairs of regular points $p, q \in L_C$, which are preimages of pairs $a, \phi(a) \in L$ in involution, are crossing a cone point.

Polygonal Billiards. There is a difficulty to construct examples of polygonal billiard along these lines: the unfolding usually interferes with the covering construction. More precisely: given an n -gon G and an unfolding UG with unfolding group H and suppose there is a covering map $\pi : UG \rightarrow \mathbb{R}^2/2\mathbb{Z}^2$, then given a point $p \in UG$ the set $F_p := \pi^{-1}(-\pi(p))$ is not always invariant under the action of H . At least we were not able to find an example for which $H \cdot F_p = F_p$. Connecting p with a point $q \in H \cdot F_p \setminus F_p$ one can fold the obtained geodesic segment to a saddle connection connecting p with a point $F_p \cap G$. This destroys examples like the swiss cross tiled by five squares.

Other applications of invariant subspaces.

The initial purpose of looking at $\mathrm{SL}(X, \omega)$ invariant subspaces $S \subset X^2$ of dimension 2 was using their nice structure as a translation surface to write down asymptotic constants in terms of the translation geometry of the invariant subspace S , like in [EMW] or in [S2, S3]. In particular we can look at modular fibers \mathcal{F} of surfaces parametrizing branched covers of X , such that there is a covering map $\pi : \mathcal{F} \rightarrow S$. Note that in such a case \mathcal{F} is a lattice surface $(\mathcal{F}, \omega_{\mathcal{F}})$ and

$$\mathrm{SL}(\mathcal{F}, \omega_{\mathcal{F}}) \cong \mathrm{SL}(S, \omega_S) \cong \mathrm{SL}(X, \omega).$$

We postpone results regarding quadratic asymptotic constants in connection with invariant subspaces of X^n to the forthcoming paper [S5]. Theorem 4 together with the other results on invariant subspaces and the asymptotic formulae described in [S2, S3] allow to evaluate asymptotic constants without using the Siegel-Veech formula directly.

9. FURTHER REMARKS AND OPEN PROBLEMS

As pointed out to us by Thierry Monteil, for a “generic” translation surface every point illuminates every point. For example, for hyperelliptic strata, Veech has shown that for almost all parameter choices one can choose a fundamental domain which is convex [V2], and thus every point illuminates every point. A similar but more technical argument works in other strata.

Alex Eskin remarked to us that our type of theorem will hold in any case to which we can generalize Ratner’s machinery.

All known examples of nonillumination in polygons use the same mechanism which leads us to state the following conjecture.

Conjecture 1: In any rational polygon P , any point p illuminates all but finitely many points $q \in P$.

Regarding our results, in particular that non-illuminable pairs are already pairs of periodic points, one might suspect that on a general translation surface, i.e. not a branched cover of a Veech surface, there are no nonilluminable configurations, unless they are (indirectly) caused by fixed points of an involution.

The convention we have taken is that orbits which arrive at corners of a polygon (singularities of a translation surface) stop. There are always (at most) two continuations possible, which we call “singular orbits or geodesics”.

Conjecture 2: Consider a translation surface (X, ω) and $p, q \in X$ such that p does not illuminate q . Then there is a “singular geodesic” connecting p to q .

It is natural to study the points which are illuminated at time t , i.e. the range of the map $\exp_p(t)$. This leads to the following questions.

Question 1: Consider a translation surface (X, ω) . For a point $p \in X$ consider the exponential map $\exp_p(t)$. For which p is the set $\exp_p(t)$ asymptotically dense? asymptotically well distributed?

Question 2: What can be said for irrational polygons?

REFERENCES

- [B] M. Boshernitzan, *Unpublished letter to H. Masur*, 1986.

- [CG] N. Chernov, G. Galperin, *Search light in billiard tables*, Regul. Chaotic Dyn. 8 (2003) 225–241.
- [CFG] H. Croft, K. Falconer, R. Guy, *Unsolved problems in geometry*, Problem Books in Mathematics. Unsolved Problems in Intuitive Mathematics, II. Springer-Verlag, New York, 1994.
- [EM] A. Eskin, H. Masur, *Asymptotic formulas on flat surfaces*. Ergodic Theory Dynam. Systems 21 (2001), no. 2, 443–478.
- [EMS] A. Eskin, H. Masur, M. Schmoll, *Billiards in rectangles with barriers*, Duke Math. J. 118 (2003), no. 3, 427–463.
- [EMW] A. Eskin, J. Marklof, D. Witte-Morris, *Unipotent flows on the space of branched covers of Veech surfaces*, preprint.
- [G] E. Gutkin, *Blocking of billiard orbits and security for polygons and flat surfaces*, Geom. Funct. Anal. 15 (2005), no. 1, 83–105.
- [GHS] E. Gutkin, P. Hubert, T. Schmidt, *Affine diffeomorphisms of translation surfaces: periodic points, Fuchsian groups and arithmeticity*, Ann. Sci. Ecole Normale Sup. 36 (2003), 847–866.
- [HS] P. Hiemer, V. Snurnikov, *Polygonal billiards with small obstacles*, J. Statist. Phys. 90 (1998) 453–466.
- [HL] P. Hubert, S. Lelivre, *Prime arithmetic Teichmüller discs in $\mathcal{H}(2)$* , To appear in Israel Journal of Math.
- [I] H. Iwaniec *On the problem of Jacobsthal*, Demonstratio Math. 11 (1978), no. 1, 225–231.
- [KH] A. Katok, B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, 54. Cambridge University Press, Cambridge, 1995.
- [K11] V. Klee, *Is every polygonal region illuminable from some point?*, Amer. Math. Monthly 76 (1969) 180.
- [K12] V. Klee, *Some unsolved problems in plane geometry*, Math. Mag. 52 (1979) 131–145.
- [KIW] V. Klee, S. Wagon, *Old and new unsolved problems in plane geometry and number theory*, The Dolciani Mathematical Expositions, 11. Mathematical Association of America, Washington, DC, 1991.
- [McM1] C. McMullen, *Teichmüller curves on Hilbert modular surfaces*, J. Amer. Math. Soc. 16 (2003), no. 4, 857–885 (electronic).
- [McM2] C. McMullen, *Teichmüller curves in genus two: Torsion divisors and ratios of sines*, preprint 2004.
- [McM3] C. McMullen, *Prym varieties and Teichmüller curves*, Preprint 2005.
- [MT] H. Masur and S. Tabachnikov, *Rational billiards and flat structures*, Handbook of dynamical systems, Vol. 1A, 1015–1089, North-Holland, Amsterdam, 2002.
- [M1] M. Möller, *Periodic points on Veech surfaces and the Mordell-Weil group of a Teichmüller curve*, preprint 2004.
- [M2] M. Möller, *Finiteness results for Teichmüller curves*, preprint 2005.
- [Mo1] T. Monteil, *A counter-example to the theorem of Hiemer and Snurnikov*, J. Statist. Phys. 114 (2004) 1619–1623.
- [Mo2] T. Monteil, *On the finite blocking property*, Annales Institut Fourier, 55 (2005) 1195–1217.

- [PP] L. Penrose and R. Penrose, *Puzzles for Christmas*, New Scientists 25 (December 1958) 1580–1581.
- [S1] M. Schmoll, *On the growth rate of saddle connections and closed geodesics on spaces of marked tori*, Geom. Funct. anal., Vol. 12 (2002) 622–649.
- [S2] M. Schmoll, *Moduli spaces of covers of Veech surfaces I: d-symmetric differentials*, preprint 2004/2005.
- [S3] M. Schmoll, *Spaces of elliptic differentials, Algebraic and topological dynamics, 303–320, Contemporary Mathematics 285, AMS 2005.*
- [S4] M. Schmoll, *in preparation.*
- [S5] M. Schmoll, *Moduli spaces of covers of Veech surfaces II: Counting and invariant subspaces*, in final preparation.
- [Ta] S. Tabachnikov, *Billiards*, “Panoramas et Synthèses”, Soc. Math. France (1995).
- [To] G. Tokarsky, *Polygonal rooms not illuminable from every point*, Amer. Math. Monthly 102 (1995) 867–879.
- [V1] W. Veech, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards*, Invent. Math. 97 (1989), no. 3, 553–583.
- [V2] W. Veech, *The billiard in a regular polygon*, Geom. Funct. Anal. 2 (1992), no. 3, 341–379.
- [V3] W. Veech, *Siegel measures*. Ann. of Math. (2) 148 (1998), no. 3, 895–944.
- [Z1] A. Zorich, *Square tiled surfaces and Teichmüller volumes of the moduli spaces of abelian differentials*. Rigidity in dynamics and geometry (Cambridge, 2000), 459–471, Springer, Berlin, 2002.
- [Z2] A. Zorich, *Flat surfaces*, preprint.

LABORATOIRE D’ANALYSE, TOPOLOGIE ET PROBABILITS, FDRATION DE RECHERCHES
DES UNITS DE MATHMATIQUES DE MARSEILLE AND, UNIVERSIT PAUL CZANNE,
CASE COUR A, AVENUE ESCADRILLE NORMANDIE-NIEMEN, F-13397 MARSEILLE
CEDEX 20, FRANCE

E-mail address: `hubert@cmi.univ-mrs.fr`

UNIVERSITY OF NOTRE DAME, DEPARTMENT OF MATHEMATICS, 255 HURLEY
HALL, NOTRE DAME, IN 46556-4618

E-mail address: `schmoll.2@nd.edu`

CENTRE DE PHYSIQUE THORIQUE, INSTITUT DE MATHMATIQUES DE LUMINY,
FDRATION DE RECHERCHES DES UNITS DE MATHMATIQUES DE MARSEILLE AND,
UNIVERSIT DE LA MDITERRANE, LUMINY, CASE 907, F-13288 MARSEILLE CEDEX
9, FRANCE

E-mail address: `troubetz@iml.univ-mrs.fr`

URL: `http://iml.univ-mrs.fr/~troubetz/`